Asymptotic behavior of quadratic differentials

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Summary

Every closed oriented surface S of genus $g \geq 2$ can be endowed with a non-unique hyperbolic metric. By the celebrated Uniformization Theorem, hyperbolic and complex structures on S are in one-to-one correspondence. A question that arises is: Can we parametrize the complex structures in a nice way? This question brings us to Teichmüller theory.

Teichmüller space is the quotient of all complex structures on S by the group of diffeomorphisms isotopic to the identity. The quotient of the Teichmüller space by the mapping class group, i.e. the group of isotopy classes of orientation preserving diffeomorphisms of S, is the moduli space, i.e. the set of all biholomorphic equivalence classes of complex structures S can be endowed with.

The moduli space classifies Riemann surfaces of a given genus up to biholomorphic equivalence. But it turns out that, instead of looking at the structure of the moduli space directly, it is easier to study the Teichmüller space: The Teichmüller space is a complex manifold biholomorphic to a bounded domain in $\mathbb{C}^{3g-3}$, whereas the moduli space is an orbifold rather than a manifold and has complicated topology. There are several mapping class group-invariant metrics on Teichmüller space which are useful when studying the structure of the moduli space and the Teichmüller space. Among them is the Teichmüller metric, a complete Finsler metric that is well-suited to measure differences in complex structures. For any pair of points in Teichmüller space there is a homeomorphism which maps one point to the other (respecting the marking) and which is quasi-conform with respect to the two complex structures. The Teichmüller metric measures how much the "optimal" quasi-conformal map differs from a biholomorphic isomorphism.

The cotangent bundle of the Teichmüller space can be identified with the bundle of holomorphic quadratic differentials. There is a one-to-one correspondence between unit speed Teichmüller geodesics and holomorphic quadratic differentials of unit area. The vertical foliations of quadratic differentials play the role of "directions" in Teichmüller space. We call a pair of geodesics with common direction asymptotic if there are unit speed parametrizations of the geodesics such that the distance between the two geodesics converges to zero.

Teichmüller space together with the Teichmüller metric is not negatively curved in any reasonable sense, but it shares many properties with negatively curved spaces. In 1980 Masur proved that for almost all directions pairs of Teichmüller geodesics in that direction are asymptotic. On the other hand, Masur proved that there are pairs of Teichmüller geodesics pointing in common direction and which stay bounded distance apart. The distance between two asymptotic geodesics in the hyperbolic plane decreases as an exponential function. To study the extend to which Teichmüller metric shows the behavior of negative curvature one can ask the question: How fast does the distance between asymptotic Teichmüller geodesics decrease?

One outcome of this thesis is that the generic pair of Teichmüller geodesics with common direction actually is exponentially asymptotic. We formulate the result as:

Theorem (Theorem 2.24 and Section 2.8)**.** *Let $d_{\mathcal{T}}$ denote the Teichmüller metric. For almost all Teichmüller geodesics ρ_1, all Teichmüller geodesics ρ_2 in the same direction are exponentially asymptotic: There are unit speed parametrizations of the*

geodesics and constants $\xi = \xi(\rho_1) > 0$ *and* $D = D(\rho_1, \rho_2) > 0$ *with*

$$d_{\mathscr{T}}\big(\rho_1(t), \rho_2(t)\big) \leq D \exp(-\xi t)$$

for large $t > 0$*. On the other hand, there are examples of slowly asymptotic pairs of Teichmüller geodesics for which such constants do not exist.*

One tool to understand Teichmüller geometry is the theory of holomorphic quadratic differentials and singular euclidean metrics. These metrics can be seen as an attempt to flatten out a hyperbolic metric to one with zero curvature. By the powerful Gauss-Bonnet formula such an attempt is bound to fail, but one can concentrate all curvature in a discrete set of singularities. For these so-called singular euclidean metrics there is a globally consistent notion of direction. As in the case of the flat torus, we get a directional foliation for every direction. One can ask for the dynamical properties of the "flow" defined by the foliation. (Off the singularities this is a flow, but it is not well-defined at the singularities. For the questions on the dynamical properties, this issue can be handled.) The dynamics in vertical direction are strongly connected to the long term behavior of Teichmüller geodesics. In the torus case, each direction is either periodic or the flow lines are dense ("minimal") and, up to scale, there is only one transverse measure invariant under the flow ("uniquely ergodic"). It is an active area of mathematical research to translate dynamical properties of the torus case to higher genus. However, the above mentioned dichotomy does not necessarily hold in higher genus. The classification of singular euclidean metrics in higher genus according to whether or not this dichotomy holds is open.
The second outcome of this thesis is a sufficient criterion for singular euclidean metrics in genus 3 not to fulfill the dichotomy:

Theorem (Theorem 3.1)**.** *If the "vertical flow" of an orientable holomorphic quadratic differential in genus* 3 *has four components, two of them cylindrical and isometric to each other and two of them minimal, then there are uncountably many minimal non-ergodic directions.*

The thesis is divided into three chapters. The first chapter provides background information needed to formulate precisely and to prove rigorously the two above mentioned theorems. Moreover, the historical context is given and the results are compared to current research status. Most of this material is well-known. References are included, whereas proofs are omitted. An exception is Section 1.3, where we give proofs fitting to our context.

The second chapter deals with the quantitative asymptotic behavior of pairs of Teichmüller geodesics. Techniques necessary for the proof are developed and a proof of the theorem is given. We rely on notations introduced in Chapter 1 and the results given in Section 1.3. Beside that, Chapter 2 is mostly self-contained.
The proof is based on the theory of zippered rectangles. Zippered rectangles are a tool to describe the geometry of quadratic differentials. Rauzy-Veech-steps change zippered rectangles according to the transformation of quadratic differentials along Teichmüller geodesics. We bound Teichmüller distance from above along asymptotic pairs of Teichmüller geodesics in terms of zippered rectangles. Under certain assumptions, Rauzy-Veech-steps reduce this bound by a uniform factor. Studying the geometry of zippered rectangles and making use of Birkhoff's Ergodic Theorem we can show that generically these assumptions are fulfilled often enough to get

exponential asymptotics.
However, exponential asymptotics do not hold in general. We construct an example of slow asymptotics, illustrating this fact.

The last chapter contains the result on dynamics of the "vertical flow" of quadratic differentials and its proof. Some familiarity with quadratic differentials given, this chapter can be read independently of the other chapters.
We use a theorem of Masur and Smillie from 1991 stating that certain sequences of directions on quadratic differentials converge to non-ergodic directions. By an inductive argument we construct uncountably many of these sequences. Not all of them have to converge to minimal directions, but we can argue that among them there certainly are uncountably many sequences converging to minimal directions. The inductive step is based on a careful study of the geometry of the given quadratic differential and uses a Ratner-style theorem.

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Contents

1. History and background information

As mentioned in the summary, this chapter contains the historic context of the two main results. Moreover, definitions and well-known results from the wide and beautiful area of Teichmüller theory are given, where we restrict to aspects related to the aim of this thesis.
The first section provides us with the necessary definitions of Teichmüller space and related objects. It follows a discussion on the curvature of Teichmüller space and on the behavior of Teichmüller geodesics. The third section establishes the relation of the dynamics on a surface and on ramified covers. We then investigate dynamical properties of special singular euclidean metrics strongly connected to Teichmüller geodesics.

1.1. **Teichmüller space and quadratic differentials.** From a topological point of view an oriented compact surface without boundaries is completely determined by its genus g. We know that there are additional structures that a surface can have. For us, complex or hyperbolic structures are of special interest. In contrast to the topology, these structures are not determined by the genus. The Uniformization Theorem states that for $g \geq 2$ there is a one-to-one correspondence between complex and hyperbolic structures. Teichmüller theory deals with these structures and their relations. Textbooks on Teichmüller theory are Imayoshi and Taniguchi [IT92] and Hubbard [Hub06], to mention just two of them.

1.1.1. *Teichmüller space and Moduli space.* Let S be an oriented compact surface without boundaries, of genus g, and with at most finitely many punctures. The topological structure uniquely determines a differentiable structure on S, which we again denote by S. We assume the Euler characteristic of S to be negative, thus S admits complete finite volume hyperbolic metrics. A Riemann surface X is a one-dimensional complex manifold. We say that X is *modeled on* S if the underlying surface is diffeomorphic to S. A *marked Riemann surface* (X, f) is a pair of a Riemann surface X modeled on S and an orientation preserving diffeomorphism $f : S \to X$, called the *marking*. Two such pairs (X_1, f_1) and (X_2, f_2) are defined to be equivalent if there exists a biholomorphic isomorphism $\Phi : X_1 \to X_2$ such that $f_2^{-1} \circ \Phi \circ f_1 : S \to S$ is isotopic to the identity on S. The *Teichmüller space* $\mathscr{T} = \mathscr{T}(S)$ of S is the set of all equivalence classes with respect to this equivalence relation. If there will be no confusion which marking is meant or if the marking itself does not matter, we sometimes just use the Riemann surface X to name a point in Teichmüller space and assume the marking being implicitly given. For $\varepsilon > 0$ define $\mathscr{T}_\varepsilon = \mathscr{T}_\varepsilon(S)$, the *thick part of Teichmüller space*, to be the subset of all $(X, f) \in \mathscr{T}(S)$ subject to the condition that the length, measured with respect to the hyperbolic metric on (X, f), of every closed curve is at least ε.

Let $\mathrm{Map}(S)$ be the *mapping class group*, i.e. the set of isotopy classes of orientation preserving diffeomorphisms on S fixing the punctures, if any. It acts on Teichmüller space by precomposition with the marking. The complex structure is not affected, only the marking changes. For a given point in Teichmüller space, its orbit consists precisely of all possible markings of the respective Riemann surface. Hence the quotient $\mathscr{M}(S)$ of Teichmüller space by the mapping class group is just the set of biholomorphic equivalence classes of Riemann surfaces modeled on S. It is called the *moduli space* of S. The image of $\mathscr{T}_\varepsilon(S)$ in moduli space is $\mathscr{M}_\varepsilon(S)$, the *thick part of moduli space*.

1.1.2. *Quadratic differentials and measured foliations.* Let S be an oriented compact surface without boundaries with at most finitely many punctures and with negative Euler characteristic. Given a Riemann surface X modeled on S we define a *quadratic differential*[1] q on X to be a holomorphic section in $T^*X \otimes_X T^*X$. In local charts $\{(U_\nu, z_\nu)\}$ we can write q as $q_\nu(z_\nu)\, dz_\nu^2$ with change of coordinates $q_\nu = q_\mu\big(\frac{dz_\mu}{dz_\nu}\big)^2$ on intersections $U_\nu \cap U_\mu$, where $dz_\nu^2 = dz_\nu \otimes_{\mathbb{C}} dz_\nu$ and q_ν is holomorphic on U_ν. Note that we allow quadratic differentials to be the square of Abelian differentials. In this case we speak of *orientable* quadratic differentials. We mention Strebel's textbook [Str84] on quadratic differentials.

From now on we assume all surfaces to be of genus $g \geq 2$ and without punctures, unless otherwise noted. A *measured foliation* $\mathcal{F}$ on S is a singular foliation on S together with a measure which is defined on transverse arcs and is invariant under isotopies along leaves of $\mathcal{F}$. We assume that every regular point of the foliation has a neighborhood in the maximal atlas of S such that $\mathcal{F}$ is induced by dy. Local neighborhoods of singular points s of $\mathcal{F}$ have a $(k(s)+2)$-pronged singularity, $k(s) > 0$. We allow $k(r) = 0$ for regular points r. By the Euler-Poincaré formula the topology of S restricts the possible singularity patterns of measured foliations: $4g - 4 = \sum_{x \in S} k(x)$. For a simple closed curve γ, denote its measure by $\mathcal{F}(\gamma)$, where $\mathcal{F}(\gamma)$ is the infimum of the transverse measures of simple closed curves in the homotopy class of γ. Fathi-Laudenbach-Poenaru [FLP79] is a good reference to measured foliations.

A *saddle connection* of a measured foliation $\mathcal{F}$ is a compact subsegment of a leaf and which starts and ends in singular points of $\mathcal{F}$. If the surface S has punctures, the punctures are treated as singular points. A *Whitehead move* is a homotopy on S collapsing one saddle connection of a foliation and the two singular points at its ends into one singular point. The equivalence relation on measured foliations generated by Whitehead moves and isotopies is called *topological equivalence*. We remark that this relation is independent of the transversal measures. Let $\mathcal{S} = \mathcal{S}(S)$ be the set of homotopy classes of simple closed curves on S. Every measured foliation $\mathcal{F}$ assigns a well-defined transversal length $\mathcal{F}([\gamma]) = \mathcal{F}(\gamma)$ to $[\gamma] \in \mathcal{S}$ and thus defines a map $\mathcal{S} \to \mathbb{R}_{\geq 0}$. Call two measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ *equivalent* if their images in $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ coincide. This equivalence relation allows Whitehead moves. It is the finest equivalence relation for measured foliations coarser than topological equivalence and with a Hausdorff set of equivalence classes (see Hubbard and Masur [HM79]). This equivalence relation endows the set $\mathcal{MF}$ of equivalence classes with a topology coming from $\mathbb{R}^{\mathcal{S}}$. Let $\mathcal{F}$ be a measured foliation on S. A *cylinder* of $\mathcal{F}$ is a maximal family of closed leaves of $\mathcal{F}$ that jointly fill up a topological annulus. The boundary of every cylinder always is a union of saddle connections. If $\mathcal{F}$ is a union of cylinders, we call $\mathcal{F}$ *periodic*. A *minimal component* of $\mathcal{F}$ is the closure of a non-compact leaf of $\mathcal{F}$. If $\mathcal{F}$ has saddle connections, the boundary of every minimal component is a union of saddle connections. If $\mathcal{F}$ has only one minimal component and this component contains all non-compact leaves, $\mathcal{F}$ is called *minimal*. It is a well-known fact that $\mathcal{F}$ is minimal if there aren't any saddle connections (see [MT99] for instance). A measured foliation is called *uniquely ergodic* if it admits only one transversal measure, up to scale.

Integration of a quadratic differential $q \neq 0$ locally along arcs gives a *flat metric* in the conformal class of X. The flat metric is a singular euclidean metric with

[1]All quadratic differentials are meant to be *holomorphic*, unless explicitly stated otherwise.

cone type singularities and distinguished vertical and horizontal direction. Off the zeros of q, transition maps are of the form $z \mapsto \pm z + c$, $c \in \mathbb{C}$, and zeros of order k give singular points with cone angle $(k+2)\pi$ in the flat metric. Vice versa, such a flat metric defines a quadratic differential via $q = dz^2$ in local charts. Transition maps preserve (unoriented) angles, hence for any direction $\theta \in [0, \pi)$ we get a measured foliation which locally is given by $\operatorname{Im} d(e^{-i\theta} z)^2$. We explicitly mention the horizontal and the vertical foliation given by $\theta = 0$ and $\theta = \pi/2$. The foliation in direction θ of q equals the horizontal foliation of $e^{-2i\theta} q$. The euclidean norm is invariant under the transition maps, thus we have a well defined notion of the length of an arc α. We denote this length by $\ell_q^*(\alpha)$ and call it the *ℓ_q^*-length* of α. *Geodesics* in the flat metric are arcs which locally minimize ℓ_q^*. Off the singularities geodesics in this metric are straight euclidean lines. At singularities geodesics are composed of straight euclidean lines which meet at the singularity and make an angle of at least π on either side. For every simple closed curve γ either there is exactly one geodesic in the homotopy class of γ or there is a one-parameter family of parallel geodesics which jointly fill an euclidean cylinder.

It is often convenient to use the 1-norm to measure the length of an arc α.

Definition. Let $\alpha : [a, b] \to X$ be a unit speed parametrization of α with respect to the singular euclidean metric, then

$$\ell_q(\alpha) = \operatorname{vert}(\alpha) + \operatorname{hori}(\alpha)$$

is the *ℓ_q-length* of α, where

$$\operatorname{vert}(\alpha) = \int_a^b |\operatorname{Re} \sqrt{q}|\big(\alpha'(t)\big)\, dt$$

is the *vertical length* and

$$\operatorname{hori}(\alpha) = \int_a^b |\operatorname{Im} \sqrt{q}|\big(\alpha'(t)\big)\, dt$$

is the *horizontal length* of α.

Remark. By the Cauchy-Schwarz-inequality we have $\ell_q^*(\alpha)/2 \le \ell_q(\alpha) \le 2\ell_q^*(\alpha)$ for every arc α (see [Raf05] for a computation of the first inequality; the other one holds by a similar argument).

Geodesics with respect to ℓ_q and with respect to ℓ_q^* do not necessarily coincide. We emphasize that in this thesis we only consider geodesics with respect to ℓ_q^*, whereas we sometimes tend to use ℓ_q to measure lengths explicitly. We apologize for any confusion that may arise.

A *saddle connection* in the flat metric defined by $q \neq 0$ is a saddle connection of a directional foliation for q. Saddle connections are geodesic arcs. If $q \neq 0$ is an orientable unit area quadratic differential and γ a saddle connection or the core curve of a cylinder in the flat metric defined by q, then γ makes a well defined angle $\theta \in [0, 2\pi)$ to the horizontal. The complex vector $\ell_q^*(\gamma)\, e^{i\theta} = \int_\gamma \sqrt{q}\, dz$ is the *holonomy* of γ.

Remark. Let γ be a non-trivial closed geodesic in the flat metric defined by $q \neq 0$. There are two cases. First, γ consists of one or more saddle connections; second, γ avoids any zero of q. In the second case, γ and its equidistant parallels fill up a flat cylinder, whose boundary circles (which have the same ℓ_q-length as γ) consist of saddle connections. Either way, there is a saddle connection with horizontal and vertical length bounded from above by the horizontal and vertical length of γ.

There is a strong connection between quadratic differentials and measured foliations, given by the Hubbard-Masur Theorem 1.1. Before we present this theorem we need to introduce a topology on Teichmüller space.

1.1.3. *Teichmüller metric and the cotangent bundle.* We endow Teichmüller space with the *Teichmüller metric*: For any two points (X_1, f_1) and (X_2, f_2) in Teichmüller space define their distance to be $d_{\mathscr{T}}\big((X_1, f_1), (X_2, f_2)\big) = \frac{1}{2}\log(\inf_\Phi K_\Phi)$, where K_Φ is the quasi-conformality constant of Φ, and Φ varies over all quasi-conformal maps $\Phi : X_1 \to X_2$ with Φ homotopic to $f_2 \circ f_1^{-1}$. Note that biholomorphic isomorphisms are conformal maps. This metric is Map-invariant and projects down to $\mathscr{M}(S)$. Teichmüller space and moduli space inherit a topology from Teichmüller metric. With this topology, Teichmüller space $\mathscr{T}(S)$ is an euclidean $(6g-6)$-dimensional open ball if S is closed, oriented, without punctures and of genus $g \geq 2$. The structure of moduli space is much more difficult; it is an orbifold rather then a manifold. Nevertheless, the thick part of moduli space is compact. *Teichmüller geodesics* are bi-infinite arcs in Teichmüller spaces which locally minimize Teichmüller distance. Let p be a point on a Teichmüller geodesic. The geodesic is divided by p into two infinite sub-arcs called *Teichmüller rays.* Compact subarcs of Teichmüller geodesics are *Teichmüller segments.*

The famous Hubbard-Masur Theorem establishes a strong relation between quadratic differentials and measured foliations:

Theorem 1.1 ([HM79])**.** *Let $\mathcal{F}$ be a measured foliation. For every point $X \in \mathscr{T}$, there is a unique quadratic differential $q(X, \mathcal{F})$ on X whose vertical foliation is equivalent to $\mathcal{F}$. The map $X \mapsto q(X, \mathcal{F})$ is a homeomorphism onto its image.*

In the spirit of this theorem we will use the pair $(X, \mathcal{F})$ to denote the quadratic differential $q = q(X, \mathcal{F})$ on X with vertical foliation equivalent to $\mathcal{F}$. We call the vertical foliation of $q(X, \mathcal{F})$ the *realization* of $\mathcal{F}$ on X.

There is a notion of *area of quadratic differentials*: $\text{area}(q) = \int_X |q(z)|dx\,dy$ with $z = x + iy$ in local charts. The area of a quadratic differential coincides with the area of the induced flat metric. Unit area quadratic differentials play a central role in this thesis.

The bundle $\Omega^2(S)$ of quadratic differentials over Teichmüller space $\mathscr{T}(S)$ is stratified by strata. A quadratic differential $q \in \Omega^2(S)$ belongs to the *stratum* $\mathcal{Q} = \mathcal{Q}(\kappa)$, $\kappa = (\kappa_1, \cdots, \kappa_k, \kappa^*) \in \mathbb{N}^k \times \{+1, -1\}$, if the orders of zeros of q form a vector equal to $(\kappa_1, \cdots, \kappa_k)$ up to permutation and $\kappa^* = +1$ for orientable quadratic differentials or $\kappa^* = -1$ for non-orientable quadratic differentials. We denote the subset of unit area quadratic differentials by $\mathcal{Q}^1 = \mathcal{Q}^1(\kappa)$. The *thick part of the stratum* is $\mathcal{Q}^1_\varepsilon = \mathcal{Q}^1_\varepsilon(\kappa) \subset \mathcal{Q}^1$, the locus of unit area quadratic differentials q with a lower bound $\varepsilon > 0$ on the ℓ^*_q-length of any saddle connection. We remark that in a fixed stratum the length of the shortest saddle connection depends continuously on the quadratic differential. The quotients of $\mathcal{Q}$, $\mathcal{Q}^1$ and $\mathcal{Q}^1_\varepsilon$ by their stabilizers in the mapping class group are denoted by $\mathscr{Q}$, $\mathscr{Q}^1$ and $\mathscr{Q}^1_\varepsilon$, and are called *(thick part of) strata over moduli space.*

Remark. There are conditions on the vector κ for $\mathcal{Q}(\kappa)$ being nonempty. One necessary condition is $\sum_{j=1}^{k} \kappa_j = 4g-4$, compare Section 1.1.2. In most, but not in all cases this condition is sufficient, see Masur-Smillie [MS93].

Although hyperbolic length and ℓ_q^*-length are quite different, we can locate the intersection of the thick part of Teichmüller space and the image of a stratum under the projection of quadratic differentials to the underlying Riemann surfaces in terms of ℓ_q^*-lengths:

Lemma 1.2. *Let $\varepsilon > 0$. There exists $c(\varepsilon) > 0$ such that for every unit area flat metric with lower bound $c(\varepsilon)$ on the length of saddle connections it holds that the underlying Riemann surface is an element of the thick part of the respective Teichmüller space.*

Proof. Let q be any unit area quadratic differential on a Riemann surface X and let there be a simple closed curve γ that is shorter than ε when measured in the hyperbolic metric σ in the given conformal class. Then the extremal length $\mathrm{ext}_{X,\sigma}(\gamma)$ is bounded from above by $c = c(\varepsilon) > 0$, where $c(\varepsilon) \searrow 0$ for $\varepsilon \to 0$, see Maskit [Mas85]. Hence, in the flat metric the length of the shortest curve in the homotopy class of γ is bounded from above by $\sqrt{c \cdot \mathrm{area}(q)} = \sqrt{c}$. Recall that the shortest saddle connection is not longer than the shortest closed curve. This finishes the proof. $\square$

The bundle $\Omega^2(S)$ of quadratic differentials can be identified with the cotangent bundle of Teichmüller space. There is a norm on the tangent bundle of Teichmüller space such that the area of quadratic differentials is the dual norm. Teichmüller metric is a Finsler metric coming from this norm on the tangent bundle. Thus unit area quadratic differentials correspond to unit tangent vectors.

Teichmüller geodesics have a very nice description in terms of flat metrics or quadratic differentials: Teichmüller geodesics in direction of a quadratic differential $q \neq 0$ are precisely the image in Teichmüller space of scaling horizontal lengths in the flat metric defined by q with $e^{t/2}$ and contracting vertical lengths by the same factor, thus Teichmüller geodesic flow is just the action of the diagonal group $\{\mathrm{diag}(e^{t/2}, e^{-t/2}) : t \in \mathbb{R}\}$ on flat metrics defined by unit area quadratic differentials. In terms of the quadratic differential this corresponds to scaling (the transversal measure of) the horizontal foliation of q by $e^{-t/2}$ and (the transversal measure of) the vertical foliation of the quadratic differential by $e^{t/2}$. Doing so we do not change area, nor do we leave the stratum. We write ρ_q to refer to the Teichmüller geodesic defined by q, normalized such that $\rho_q(0) = X$. For a unit area quadratic differential q on X, the parametrization coming from the $\mathrm{diag}(e^{t/2}, e^{-t/2})$-action turns $\{\rho_q(t) : t \in \mathbb{R}\}$ into a unit speed Teichmüller geodesic. Let q_t be the unit area quadratic differential on $X_t = \rho_q(t)$ that gives rise to the Teichmüller geodesic ρ_q, up to parametrization. With this convention, $(X_0, q_0) = (X, q)$ is the pair of the Riemann surface X and the quadratic differential q on that surface.

Definition. Let $q \in \mathcal{Q}^1$ be a unit area quadratic differential on $X \in \mathscr{T}(S)$ and let $\varepsilon > 0$ be given. We say that the a point $\rho_q(t)$ on the Teichmüller geodesic is in the thick part $\mathcal{Q}^1_\varepsilon$ of the stratum if the corresponding unit area quadratic differential is in the thick part: $q_t \in \mathcal{Q}^1_\varepsilon$.

Given $X \in \mathscr{T}(S)$ and a measured foliation $\mathcal{F}$, we use $\rho_{X,\mathcal{F}}$ to denote the Teichmüller geodesic $\rho_{q(X,\mathcal{F})}$ defined via the Hubbard-Masur Theorem 1.1.

Remark. For every $X^* \in \mathscr{T}$ and every measured foliation $\mathcal{F}$ there exists a unique $X \in \mathscr{T}$ such that $(X, \mathcal{F})$ has unit area and that the two Teichmüller geodesics $\rho_{X^*,\mathcal{F}}$ and $\rho_{X,\mathcal{F}}$ are the same, up to parametrization.

Considering quadratic differentials as cotangent vectors, Earle [Ear77] proved that Teichmüller distance is continuously differentiable off the diagonal in the following sense: Fix $X \in \mathscr{T}(S)$ and let $D_X(Y) = d_{\mathscr{T}}(X, Y)$. The map $D_X : \mathscr{T}(S) \setminus \{X\} \to \mathbb{R}$ is continuously differentiable. (Earle uses another parametrization of Teichmüller rays, not proportional to arc length). Veech [Vee90] gave local complex coordinates for each stratum of orientable quadratic differentials.

Teichmüller space with Teichmüller metric is a geodesic metric space, i.e. for any pair of points there exists a geodesic realizing the distance; in fact there exists exactly one length-minimizing geodesic between two points in Teichmüller space. Moreover, $\mathscr{T}(S)$ is complete as a metric space, but it is not non-positively curved[2]. Teichmüller metric is not the only interesting metric on Teichmüller space; the Weil-Peterson metric for instance turns Teichmüller space into a non-positively curved but non-complete metric space, see Hubbard's textbook [Hub06]. Throughout this thesis we only consider Teichmüller space with the Teichmüller metric.

Convention. From now on, we assume that every quadratic differential is non-vanishing.

1.2. **Teichmüller geometry: Curvature and geodesics.** For any Teichmüller geodesic ρ there is a unique complexification of ρ which is an isometrically embedded image of the Poincaré disc in $\mathscr{T}(S)$ and which contains ρ. This image is called *Teichmüller disc* and the restriction of the Teichmüller metric to any Teichmüller disc is negatively curved. In the late 1950's the first results on the curvature of Teichmüller space were published. Teichmüller space with Teichmüller metric was believed to be of negative curvature. This turned out to be false. Beginning with the 1970's new results on the curvature of Teichmüller space were established.

1.2.1. *Curvature.* Teichmüller space is not non-positively curved in the sense of Busemann[3]. This is due to Masur [Mas75]. He showed that two different Teichmüller rays $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ not in the same Teichmüller disc stay in bounded distance if the vertical measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ decompose into cylinders and if the homotopy classes of the cylinders of $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide, i.e. if the foliations are topologically equivalent and without non-compact leaves. On the other hand, Teichmüller space and moduli space share some properties with negatively curved spaces. In most cases converging pairs of Teichmüller rays are asymptotic: A result of Masur's is that pairs of Teichmüller rays whose quadratic differentials have common uniquely ergodic vertical foliation are positively asymptotic ([Mas80]). Moreover, the Teichmüller geodesic flow on moduli space is ergodic (equally due to Masur, [Mas82a]). These phenomena could possibly arise from Teichmüller space being negatively curved in a weaker sense. Masur and Wolf [MW95] answered

[2]We will come back to this fact in the following.

[3]A space is non-positively curved in the sense of Busemann if the distance between the endpoints of two geodesic segments emanating from the same point is at least as large as twice the distance between the two midpoints.

the question whether or not Teichmüller space is Gromov-hyperbolic[4]: Teichmüller space is not Gromov-hyperbolic. Thus Teichmüller space has an interesting geometry which is not globally comparable to spaces of non-positive curvature, nevertheless there are isometrically embedded Poincaré discs in Teichmüller space.

In fact, the complement of the thick part shares properties of spaces with positive curvature. Minsky ([Min96]) found a very nice description of the Teichmüller metric outside the thick part: For $\varepsilon > 0$ and a *multicurve* Γ (i.e. a set of disjoint simple closed curves) let $\mathrm{Thin}_\varepsilon(\Gamma)$ be the set of all $(X, f) \in \mathscr{T}$ such that on (X, f) the hyperbolic length of every $\gamma \in \Gamma$ is at most ε. Define $Y_\Gamma = \mathscr{T}(S_\Gamma) \times \prod_{\gamma\in\Gamma} \mathbb{H}_\gamma$, where $S_\Gamma = S \setminus \Gamma$ is considered as a punctured surface and each $\mathbb{H}_\gamma$ is a copy of the hyperbolic plane. The Fenchel-Nielsen coordinates give rise to a homeomorphism $\mathscr{T}(S) \to Y_\Gamma$, where for each $\gamma \in \Gamma$ the pair $(\mathrm{twist}, \mathrm{length})$ is mapped to $(x, y) = (\mathrm{twist}, \mathrm{length}^{-1})$ in the factor $\mathbb{H}_\gamma$. Endow Y_Γ with the sup-metric. Minsky's product regions theorem states that this homeomorphism is a quasi-isometry:

Theorem 1.3 ([Min96])**.** *For ε sufficiently small, $\mathrm{Thin}_\varepsilon(\Gamma)$ and Y_Γ are isometric up to a bounded additive error.*

In spaces with a Gromov-hyperbolic metric, given $\delta > 0$ and three points x, y and z with $d(x, z)+d(z, y)-d(x, y) < \delta$, the distance $d([x, y], z)$ between a shortest path $[x, y]$ from x to y and the point z is bounded from above by a number $R(\delta)$, independent of the three points. In spaces with positive curvature, it is possible that the distance $d([x, y], z)$ grows proportional to $d(x, y)$. This happens in Minsky's product regions, too ([Min96]). In this sense, Teichmüller space has properties of positive curvature on large scales.

On the other hand, the thick part of Teichmüller space can be compared to a Gromov-hyperbolic space. Harvey [Har81] introduced the curve graph $\mathcal{C}(S)$. This graph can be made into a metric space: The vertices of $\mathcal{C}(S)$ correspond to the homotopy classes of simple closed curves on S, and there is an edge of euclidean length 1 between two vertices if the corresponding simple closed curves can be realized disjointly. Endowed with this metric, the curve graph is Gromov-hyperbolic (this is due to Masur and Minsky [MM99], also compare Bowditch [Bow06] and Hamenstädt [Ham07]). The connection between the curve graph and the thick part of Teichmüller space is as follows: Every Teichmüller geodesic projects to an unparametrized quasi-geodesic in the curve graph ([MM99]), and the image is a parametrized quasi-geodesic if and only if the Teichmüller geodesic is completely contained in a thick part of Teichmüller space. This vague statement is borrowed from [Ham07], the precise version is in Hamenstädt's paper [Ham10].

Thus Teichmüller space has regions where Teichmüller metric has properties of positive curvature, and there are regions which seems to have negative curvature. Another way to understand the geometry of Teichmüller space, different from comparing to spaces of well-known geometry, is to understand the behavior of Teichmüller geodesics.

1.2.2. *Teichmüller geodesics.* As already noted, Teichmüller geodesics can easily be described in terms of quadratic differentials. We begin our study of Teichmüller geodesics by understanding the relation of properties of a geodesic in Teichmüller space and the geometry of the respective quadratic differentials.

[4]A space is Gromov-hyperbolic if there exists a $\delta > 0$ such for every triangle the third side is contained in the δ-neighborhood of the first two sides.

Let ρ be a unit speed Teichmüller geodesic which projects to a closed geodesic in moduli space[5]. The one-parameter family of unit tangent vectors along the closed geodesic defines a loop in a stratum over moduli space, thus there is a lower bound for the length of the shortest saddle connection of any quadratic differential on that loop. In particular the vertical foliation is without saddle connections, hence minimal. Moreover, the vertical foliation is uniquely ergodic by Masur's criterion:

Theorem 1.4 ([Mas92])**.** *Every* recurrent *Teichmüller geodesic (i.e. it returns to the thick part of Teichmüller space after arbitrarily large times) is defined by quadratic differentials with uniquely ergodic vertical foliation.*

This result has been improved by Cheung and Eskin. In [CE07a] it is established that the vertical foliation is uniquely ergodic if the corresponding Teichmüller geodesic does not leave larger and larger thick parts too fast. However, closed geodesics in moduli space may avoid arbitrarily large regions inside moduli space; again in contrast to the behavior Riemann surfaces with finite volume hyperbolic metrics show. Let S be not one of three exceptional surfaces. For every $\varepsilon > 0$, Hamenstädt [Ham05] established the existence of a closed geodesic in moduli space which avoids the thick part $\mathscr{Q}_\varepsilon(S)$.

Contrary to the case of Teichmüller geodesics with closed projection to moduli space is the case of a Teichmüller geodesics whose vertical foliation has a cylinder: The core curve is pinched along the Teichmüller geodesic, and the projection of the Teichmüller geodesic to moduli space enters directly into a cusp.

These two situations are the extremal behaviors of Teichmüller geodesics, and there are Teichmüller geodesics with a behavior in between. Cheung and Eskin [CE07b] gave examples of Teichmüller geodesics with minimal non-ergodic vertical foliation which diverge to infinity arbitrarily slowly: These Teichmüller geodesics avoid arbitrarily slowly growing thick parts.

As one might already have noted, it is a general principle that the vertical foliation of the quadratic differential determines the long term behavior of the Teichmüller geodesic. This principle will be present in the study of pairs of Teichmüller geodesics, too.

1.2.3. *Pairs of Teichmüller geodesics.* In spaces with a hyperbolic metric, the distance between two converging geodesics behaves like an exponentially decreasing function. Do pairs of converging Teichmüller geodesics have the same property? To answer this question, we first classify pairs of Teichmüller geodesics into asymptotic pairs, diverging pairs, and pairs which stay asymptotically in finite distance.

An easy observation is the following. Pairs of Teichmüller rays inside the same Teichmüller disc behave exactly as hyperbolic geodesics, since Teichmüller discs are isometrically embedded hyperbolic planes in Teichmüller space.

Suppose that we have two Teichmüller rays ρ_1 and ρ_2 whose vertical foliations[6] $\mathcal{F}_1$ and $\mathcal{F}_2$ are not topologically equivalent. The rays diverge in the sense that $d_{\mathscr{T}}\big(\rho_1(t), \rho_2(t)\big) \to \infty$ as $t \to \infty$. The case of nonzero intersection number $i(\mathcal{F}_1, \mathcal{F}_2) \neq 0$ is due to Ivanov [Iva01], Lenzhen and Masur [LM10] established the result in the case $i(\mathcal{F}_1, \mathcal{F}_2) = 0$.

[5]Closed geodesics in moduli space are in one-to-one correspondence with so-called *pseudo-Anosov*-elements in the mapping class group.

[6]The vertical foliation of a Teichmüller ray is understood to be the vertical foliation of a quadratic differential defining the ray. As a measured foliation it is defined up to scalar multiple.

Now consider the situation of topologically equivalent vertical foliations.
Let Γ be a multicurve. If a pair of Teichmüller rays is determined by a pair of measured foliations each of which decomposes the surface into a union of cylinders in the homotopy class of Γ, then the asymptotic distance along the Teichmüller geodesics is bounded from above. If additionally the rays do not belong to the same Teichmüller disc, the asymptotic distance is bounded from below, too (Masur, [Mas75] and [Mas80]).
Again, suppose that the non-uniquely ergodic measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent, but this time with at least one minimal component. There are two cases. Case one, the measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are absolutely continuous with respect to each other in the minimal components. Pairs of Teichmüller rays with vertical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ stay bounded distance apart (Ivanov [Iva01]). Case two, the two foliations are not absolutely continuous with respect to each other in at least one minimal component. Lenzhen and Masur [LM10] proved that pairs of quadratic differentials with vertical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ determine divergent Teichmüller rays.
Finally, let $\mathcal{F}$ be a uniquely ergodic measured foliation. All topologically equivalent measured foliations are of the form $\lambda\mathcal{F}$ for some $\lambda > 0$. It is a result of Masur's that pairs of Teichmüller geodesics with common vertical foliation $\mathcal{F}$ are asymptotic ([Mas80]).

Masur's result does not tell us how fast the distance between two asymptotic Teichmüller rays decreases. The thick part of Teichmüller space shows properties of negatively curved spaces. Wishful thinking suggests that the distance along asymptotic pairs of Teichmüller geodesics with common uniquely ergodic vertical foliation decreases exponentially fast if both geodesics are completely contained in the thick part of Teichmüller space. Very recently Rafi showed that, if two Teichmüller geodesic segments start and end in the thick part of Teichmüller space and with endpoints of small distance, they stay uniformly close to each other, even if the segments are not completely contained in the thick part of Teichmüller space (in preparation). This suggests that exponential asymptotics for pairs of Teichmüller rays in thick part could lead to exponential asymptotics for pairs of recurrent Teichmüller rays. However, Rafi's result relies on estimating extremal lengths of simple closed curves by calculating their lengths in the flat metrics given by the quadratic differentials along the Teichmüller geodesic segment. Comparing extremal lengths to flat metric lengths always produce multiplicative errors (see Rafi's papers [Raf05] or [Raf07], to give two examples). These errors come from Minsky's expanding cylinders ([Min92]) and translate by Kerckhoff's theorem, which relates Teichmüller distance to the logarithm of quotients of extremal lengths ([Ker80]), to additive errors in estimates for Teichmüller distance. Obviously, additive errors produce great difficulties for proving results on asymptotic rays.

With different arguments, we prove in Chapter 2 that the distance along asymptotic pairs decreases like an exponential function:

Theorem 1.5. *For almost all*[7] *pairs ρ_1 and ρ_2 of Teichmüller rays with common uniquely ergodic vertical foliation there are unit speed parametrizations of the rays with*

$$\log d_{\mathcal{T}}\big(\rho_1(t), \rho_2(t)\big) \stackrel{.}{\prec} -t$$

[7] This holds for an uncountable family of measures.

for large $t > 0$. On the other hand, there are examples of pairs of Teichmüller geodesics with common uniquely ergodic vertical foliation and slow asymptotics such that the above conclusion fails.

Here and throughout the thesis, we use the following short-form notations: If there exists $\alpha > 0$ independent of A and B, we write $A \dot{\prec} B$ for $A \leq \alpha B$. If $A \dot{\prec} B$ and $B \dot{\prec} A$ hold, we write $A \dot{\asymp} B$.

For a precise statement on the dependencies of the constants involved in this theorem we refer to Section 2.7.

1.3. **Covers and quadratic differentials.** In this section the surface S is allowed to have finitely many punctures. Let $\mathcal{F}$ be a measured foliation on $X \in \mathscr{T}(S)$ and let $p : \hat{X} \to X$ be a ramified covering with $0 < k < \infty$ sheets and such that every ramification point is either a singularity of $\mathcal{F}$ or a puncture of S. Let $\hat{\mathcal{F}} = p^* \mathcal{F}$ be the lift of $\mathcal{F}$ to $\hat{X}$.

Suppose γ is a saddle connection of $\mathcal{F}$. Its preimage in $\hat{\mathcal{F}}$ is a union of at most k saddle connections $\hat{\gamma}_i$. Vice versa, saddle connections $\hat{\gamma}$ on $\hat{\mathcal{F}}$ project to saddle connections γ on $\mathcal{F}$. The same is true if we replace 'saddle connections' by 'regular leaves of $\mathcal{F}$ which are simple closed curves'.

Lemma 1.6. *Let the covering be as above. Minimal components downstairs lift to unions of minimal components upstairs, and minimal components upstairs project to minimal components downstairs. Non-uniquely ergodic foliations do not lift to uniquely ergodic foliations.*

Proof. The statements on minimality are immediate from the discussion above.
To see the last statement, let the foliation downstairs be non-uniquely ergodic. We can find a regular transverse arc downstairs with the property that there are two measures, transverse to the foliation, which are mutually singular on that arc. The same is true for the lifted measures with respect to one preimage of the arc, hence the lifted foliation is not uniquely ergodic. □

Let $q \in \mathcal{Q}^1(S)$ be a unit area quadratic differential on $X \in \mathscr{T}(S)$. The area of the lift p^*q of q to $\hat{X}$ is k-times the area of q. Rescaling of a quadratic differential does not change the underlying Riemann surface, nor does it change the unparametrized Teichmüller geodesic. Let $\hat{q}$ be the unit area quadratic differential in the projective class of p^*q.

A special case occurs if every critical point of the covering is a zero of q or a puncture of S. In particular this occurs if a directional foliation of q equals $\mathcal{F}$. The quadratic differentials q and $\hat{q}$ define unit area flat metrics on X and $\hat{X}$. In terms of these metrics the covering projection is of the form $z \mapsto z/\sqrt{k}$ locally at regular points (mind the scaling!). At ramification points with ramification index $l \leq k$ the projection locally has the form $z \mapsto z^{1/l}/\sqrt{k}$.

Let γ be a saddle connection for q. Its lift to $\hat{q}$ is a union of at most k saddle connections $\hat{\gamma}_i$ with length $\ell^*_q(\gamma)/\sqrt{k} \leq \ell^*_{\hat{q}}(\hat{\gamma}_i) \leq \sqrt{k}\, \ell^*_q(\gamma)$. Vice versa, a saddle connection $\hat{\gamma}$ upstairs projects to a saddle connection γ downstairs, and their lengths are related by $\ell^*_{\hat{q}}(\hat{\gamma})/\sqrt{k} \leq \ell^*_q(\gamma_i) \leq \sqrt{k}\, \ell^*_{\hat{q}}(\hat{\gamma})$.

Lemma 1.7. *Let $\varepsilon > 0$ be given and let the covering be as above. If $q \in \mathcal{Q}^1_\varepsilon$ is on the boundary of the thick part of the stratum, i.e. there is a saddle connection of length exactly ε, then $\varepsilon/\sqrt{k}$ is a lower bound and $\sqrt{k}\,\varepsilon$ is an upper bound for* *the length of the shortest saddle connection on $\hat{q}$. Hence the Teichmüller geodesic defined by q returns to the thick part of the stratum after arbitrarily large times if and only if the Teichmüller geodesic defined by $\hat{q}$ does so.*

Proof. The statement on the lengths of saddle connections is established above. The last statement follows from Lemma 1.2 as the action of the Teichmüller flow $\mathrm{diag}(e^{t/2}, e^{-t/2})$ is linear and the lengths of saddle connections in the flat metrics defined by q and $\hat{q}$ differ only by a uniformly bounded multiplicative error. $\square$

Corollary 1.8. *Let $(X, \mathcal{F})$ give rise to a Teichmüller geodesic such that (X_t, q_t) returns to the thick part of the stratum after arbitrarily large times. Let $p : \hat{X} \to X$ be defined with respect to $\mathcal{F}$. Then $\mathcal{F}$ and $\hat{\mathcal{F}}$ both are uniquely ergodic measured foliations.*

Proof. As (X_t, q_t) returns to the thick part of the stratum after arbitrarily large times, there isn't any saddle connection in vertical direction, hence $\mathcal{F}$ does not have any saddle connection. Therefore $\mathcal{F}$ is minimal, and so is $\hat{\mathcal{F}}$ by Lemma 1.6. Moreover, Lemma 1.7 tells us that the lifted Teichmüller geodesic returns to the thick part of the stratum after arbitrarily large times. Lemma 1.2 implies that downstairs and upstairs recurrence takes place in the thick parts of the respective Teichmüller spaces. Now Masur's criterion (Theorem 1.4) proves the corollary. $\square$

Theorem 1.5 is a statement on quadratic differentials with uniquely ergodic measured foliations, orientable or not. The techniques we use to prove it in Chapter 2 rely on oriented measured foliations. Thus non-orientable measured foliations, which possibly arise as the vertical foliations of non-orientable quadratic differential, need extra work. We introduce a tool to solve this issue.

Let $\mathcal{F}$ be a measured foliation. If it is not orientable, we can pass to a special two sheeted cover of the surface such that the lift of $\mathcal{F}$ is orientable. Lanneau gave an explicit construction of an orientable ramified double cover.

Theorem 1.9 ([Lan04]). *Let S be compact without boundary. Let $\mathcal{F}$ be a measured foliation on S and let $X \in \mathscr{T}$ be a point in Teichmüller space such that $(X, \mathcal{F})$ is not orientable. There exists a ramified two-sheeted covering $p : \mathrm{cov}_{\mathcal{F}}(X) \to X$ such that the lift $\mathrm{cov}_{\mathcal{F}}(\mathcal{F}) = p^* \mathcal{F}$ of the foliation and the lift $\big(\mathrm{cov}_{\mathcal{F}}(X), \mathrm{cov}_{\mathcal{F}}(\mathcal{F})\big)$ of the quadratic differential become orientable and the areas of $(X, \mathcal{F})$ and $\big(\mathrm{cov}_{\mathcal{F}}(X), \mathrm{cov}_{\mathcal{F}}(\mathcal{F})\big)$ coincide. The ramification points are exactly the singular points of $\mathcal{F}$ that have an odd number of prongs. If we describe p in terms of the respective flat metrics, it locally has the form $z \mapsto z/\sqrt{2}$ at regular points of p and $z \mapsto z^{1/2}/\sqrt{2}$ at critical points of p. We call this cover the* orientation double cover*.*

Remark. This construction is a ramified covering as in the beginning of this section. Thus the above lemmas apply.

The orientation double cover depends on the foliation $\mathcal{F}$, and even its topological type is not uniquely defined for a fixed $X \in \mathscr{T}$. This follows from Theorem 1.1 for instance.

It is possible for $X \in \mathscr{T}_\varepsilon$ in the thick part of Teichmüller space that the orientation double cover carries a simple closed curve that is arbitrarily short when measured in the flat metric: If $(X, \mathcal{F})$ has a saddle connection γ that connects two odd-order singularities, this saddle connection lifts to a closed curve of at most $\sqrt{2}$ times the length of γ. Hence if we consider a sequence $X_j \in \mathscr{T}_\varepsilon$ such that on every

$(X_j, \mathcal{F})$ there is such a saddle connection γ_j with length $1/j$, the sequence of orientation double covers comes with a sequence of closed curves, the $\ell^*_{(X_j,\mathcal{F})}$-lengths of which converge to zero.

Although the orientation double cover does not necessarily preserve the thick part of Teichmüller space, it preserves Teichmüller distance:

Proposition 1.10. *Let $\mathcal{F}$ be a non-orientable measured foliation on S and let X and $\tilde{X}$ be two different points in Teichmüller space $\mathscr{T}(S)$. The distance between X and $\tilde{X}$ coincides with the distance between their orientation double covers:*

$$d_{\mathscr{T}}(X, \tilde{X}) = d_{\mathscr{T}}(\mathrm{cov}_{\mathcal{F}}(X), \mathrm{cov}_{\mathcal{F}}(\tilde{X})).$$

Note that we use the symbol $d_{\mathscr{T}}$ for the Teichmüller distance in two different Teichmüller spaces.

Proof. Let q be the unit area quadratic differential on X which defines the Teichmüller geodesic from X to $\tilde{X}$ and let $D = d_{\mathscr{T}}(X, \tilde{X})$. We want to emphasize that there is no correspondence at all between q and $\mathcal{F}$.

The lift of any unit area quadratic differential q' on any $X' \in \mathscr{T}(S)$ to a quadratic differential on $\mathrm{cov}_{\mathcal{F}}(X')$ has area twice the area of q'. Rescaling of a quadratic differential neither changes the Riemann surface nor does it change the unparametrized Teichmüller geodesic. Let $\mathrm{cov}_{\mathcal{F}}(q')$ be the unit area quadratic differential in the projective class of the lift of q'.

Recall that Teichmüller spaces are geodesic metric spaces with a unique length minimizing geodesic between any two points, and that this geodesics are described by unit area quadratic differentials. Thus it suffices to show that $\mathrm{diag}(e^{D/2}, e^{-D/2})$ transforms the flat metric given by $\mathrm{cov}_{\mathcal{F}}(q)$ on $\mathrm{cov}_{\mathcal{F}}(X)$ into a flat metric in the conformal class of $\mathrm{cov}_{\mathcal{F}}(\tilde{X})$.

From the explicit description of the orientation double cover in Theorem 1.9 we deduce that horizontal and vertical lengths in the flat metrics defined by q upstairs and $\mathrm{cov}_{\mathcal{F}}(q)$ downstairs coincide up to multiplication by $\sqrt{2}$. This commutes with the stretching/contracting-action of Teichmüller flow in the flat metric picture. As the flat metrics defined by q on X and by q_D on $X_D = \tilde{X}$ differ by multiplication with $\mathrm{diag}(e^{D/2}, e^{-D/2})$, the same is true for the lifted flat metrics. Thus $\mathrm{diag}(e^{D/2}, e^{-D/2})$ transforms the flat metric on $\mathrm{cov}_{\mathcal{F}}(X)$ into a flat metric on $\mathrm{cov}_{\mathcal{F}}(\tilde{X})$. □

1.4. **Directional foliations on quadratic differentials.** The general principle that the vertical foliations determine the long term behavior of (pairs of) Teichmüller geodesics underlines the importance of understanding the directional foliation of quadratic differentials.

1.4.1. *$SL(2,\mathbb{R})$-action on quadratic differentials.* Let $P \subset \mathbb{R}^2$ be a polygon whose sides come in pairs of equally long edges with common angle to the horizontal. We identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way. Glueing the boundary of P along paired edges by maps of the form $z \mapsto \pm z + c$, $c \in \mathbb{C}$, one gets a flat metric and a quadratic differential via dz^2 (Zorich [Zor06]). Vice versa, given $q \in \mathcal{Q}(S)$ one can find a finite set of saddle connections with disjoint interior such that by cutting along these saddle connections one decomposes the flat metric into a finite union of polygons with a pairing on the edges as above. The usual $\mathrm{GL}(2,\mathbb{R})$-action on $\mathbb{R}^2$ translates to an action on $\mathcal{Q}(S)$. The action of $\mathrm{SL}(2,\mathbb{R})$ is area preserving and the rotation subgroup $\mathrm{SO}(2)$ fixes the underlying Riemann surface. The diagonal subgroup $\{\mathrm{diag}(e^{t/2}, e^{-t/2}) : t \in \mathbb{R}\} < \mathrm{SL}(2,\mathbb{R})$ corresponds to Teichmüller flow. Its complexification is a Teichmüller discs in $\mathscr{T}(S)$ and equals the isometric projection of the orbit of a quadratic differential under $\mathbb{H} = \mathrm{SO}(2)\backslash\mathrm{SL}(2,\mathbb{R})$ to Teichmüller space. Given a quadratic differential $q \in \mathcal{Q}$, the *Veech group* $\mathrm{SL}(q) = \{A \in \mathrm{SL}(2,\mathbb{R}) : q \text{ and } A.q \text{ project to the same point in } \mathscr{Q}(S)\}$ is the stabilizer under the $\mathrm{SL}(2,\mathbb{R})$-action of the singular euclidean metric defined by q. Hence the image of $\mathrm{SO}(2)\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(q)$ in $\mathscr{M}(S)$ is isometrically embedded. If this quotient has finite volume, the flat metric defined by q is called a *Veech surface*. By abuse of notation we then call q a Veech surface, too. Veech surfaces have particularly nice dynamics, as we will see.

1.4.2. *Dynamics on flat metrics.* We take the once-punctured torus as an example to begin with. Its flat metric has very nice dynamics. It is well-known that the directional foliation in direction $\theta \in [0, \pi)$ decomposes into periodic cylinders if θ is a rational multiple of π. If θ is not a rational multiple of π, there aren't any saddle connections, hence the foliation is minimal. Moreover, it is a uniquely ergodic measured foliation. In general we say that a quadratic differential or a flat metric fulfills the *Veech dichotomy* if for every direction the directional foliation is either periodic or minimal and uniquely ergodic. We will speak of a *periodic, minimal* or *uniquely ergodic direction* if its directional foliation has the respective property.

Flat metrics which are ramified coverings of the once-punctured torus, where the ramification takes place at the puncture, are called *arithmetic surfaces*. By Section 1.3, arithmetic surfaces fulfill the Veech dichotomy. The Veech group of an arithmetic surface is commensurable to $\mathrm{SL}(2,\mathbb{Z})$, i.e. the Veech group and $\mathrm{SL}(2,\mathbb{Z})$ share a common finite index subgroup.

Arithmetic surfaces aren't the only examples of surfaces fulfilling the Veech dichotomy. It is a theorem of Veech [Vee89] that every Veech surface fulfills the Veech dichotomy. Whether or not the converse is true was an open question for some years. McMullen proved that for quadratic differentials with orientable vertical foliation in genus 2, Veech surfaces are exactly the ones fulfilling the Veech dichotomy ([McM05], see also Calta [Cal04]). In higher genus, the converse to Veech's theorem is not true. Smillie and Weiss [SW08] produced a family of quadratic differentials which fulfill the Veech dichotomy and still aren't Veech surfaces. These examples are ramified covers of Veech surfaces and are of genus at least 5. A similar construction (which originates in Hubert and Schmidt's paper [HS04]) is used by Cheung, Hubert and Masur [CHM08] to construct quadratic differentials in higher genus which satisfy a topological version of the Veech dichotomy (every direction is either periodic or minimal), but which do not satisfy Veech dichotomy itself. There aren't any quadratic differentials with this property in genus 2 ([McM05]).

Independent of the Veech dichotomy, we know more about the dynamical properties of directional foliations of quadratic differentials. As there are only countably many saddle connections in any flat metric, there are at most countably many non-minimal directions for a given flat metric. Even more is true: For every quadratic differential, the set of directions with uniquely ergodic directional foliation is dense in the unit circle and has full measure (Kerckhoff, Masur and Smillie [KMS86]). In his paper [Mas92], Masur proved that for every quadratic differential the Hausdorff dimension of the set of non-ergodic directions is at most 1/2. On the other hand, Masur and Smillie proved in their remarkable paper [MS91] the following: Up to a small number of exceptional strata over moduli space, for almost every quadratic differential in a fixed connected component[8] of the stratum, the Hausdorff dimension of the set of non-ergodic directions is bounded from below. The non-ergodicity criterion established by Masur and Smillie was used by Cheung and Masur [CM06] to prove that for every orientable genus-2 quadratic differential which is not a Veech surface, there are uncountably many minimal non-ergodic directions. The next step in classifying the dynamics of flat metrics would be to consider non-orientable quadratic differentials in genus 2. Let $q' \in \mathscr{Q}(1,1,1,1,-1)$ be a non-orientable quadratic differential in the principal stratum in genus 2 and let $\mathscr{Q}^*$ be the stratum of *meromorphic* quadratic differentials on $\mathbb{CP}^1$ with exactly two simple zeros and six simple poles. By a two-sheeted covering-construction over the Riemann sphere, ramified exactly at the simple poles of a meromorphic quadratic differential in $\mathscr{Q}^*$ (this is not the orientation double cover, as there are still simple zeros left) we get a $\mathrm{GL}(2,\mathbb{R})$-equivariant isomorphism between $\mathscr{Q}(1,1,1,1,-1)$ and $\mathscr{Q}^*$ which respects the dynamics on the flat metrics (c.f. Section 1.3 and Lanneau [Lan04]). This isomorphism maps q' to a meromorphic quadratic differential $q^* \in \mathscr{Q}^*$ with the same dynamical properties as q'. Denote the vertical foliation of q^* by $\mathcal{F}$. Now consider the orientation double cover with respect to $\mathcal{F}$. This is a two-sheeted cover ramified exactly at the poles and the zeros. The same arguments as above apply. The lift $q = \mathrm{cov}_{\mathcal{F}}(q^*)$ is an element of $\mathscr{Q}(4,4,+1)$ with the same dynamical properties as q^*, hence with the same dynamical properties as q'. The hyperelliptic involution is an element of the Veech group $\mathrm{SL}(q)$ and fixes the two zeros of q by construction. The *hyperelliptic locus* $\mathscr{L}$ of the non-hyperelliptic component $\mathscr{Q}(4,4,+1)^{odd} \subset \mathscr{Q}(4,4,+1)$ (which is the set of all orientable genus-3 quadratic differentials with two zeroes of order four and with odd spin structure[9]) is the subset of all quadratic differentials $q \in \mathscr{Q}(4,4,+1)^{odd}$ such that the underlying Riemann surface is a branched cover of the Riemann sphere and the hyperelliptic involution is an element of the Veech group $\mathrm{SL}(q)$. The hyperelliptic locus $\mathscr{L}$ is the $\mathrm{GL}(2,\mathbb{R})$-equivariantly isomorphic preimage in $\mathscr{Q}(4,4,+1)$ of $\mathscr{Q}^*$ under the double covering construction. Thus classifying the dynamics of quadratic differentials in $\mathscr{Q}(1,1,1,1,-1)$ can be done by classifying the dynamics of quadratic differentials in $\mathscr{L}$.

In Chapter 3 we prove

Theorem 1.11. *If the vertical foliation of an orientable quadratic differential in $\mathscr{L}$ decomposes as in Figure 8 into two minimal components and two periodic cylinders and if the cylinders are interchanged by the hyperelliptic involution, then there are uncountably many minimal non-ergodic directions.*

[8]More on connected components of strata is in Kontsevich's and Zorich's paper [KZ03] and in Lanneau's paper [Lan04], to mention just two sources.

[9]We do not need properties of the spin structure explicitly. A definition can be found in [KZ03], for example.

2. Exponentially attractive Teichmüller rays

Throughout this chapter we assume the underlying topological surfaces to be compact without boundaries and without any punctures. Unless otherwise stated, all measured foliations are assumed to be minimal and realized without any saddle connection. As mentioned earlier, in this chapter we will prove that the distance along asymptotic pairs decreases like an exponential function. In Section 1.2.3 we formulated this results as

Theorem 2.1. *For almost all*[10] *pairs ρ_1 and ρ_2 of Teichmüller rays with common uniquely ergodic vertical foliation there are unit speed parametrizations of the rays with*

$$\log d_{\mathscr{T}}(\rho_1(t), \rho_2(t)) \dot{\prec} -t$$

for large $t > 0$. On the other hand there are examples of pairs of Teichmüller geodesics with common uniquely ergodic vertical foliation and slow asymptotics, such that the above conclusion fails.

A precise version of the first part is Theorem 2.24 in Section 2.7. This generic result is the best one can hope for: There are asymptotic pairs of Teichmüller geodesics with common uniquely ergodic vertical foliation and arbitrarily slow asymptotics (Section 2.8).

To give an overview of how this chapter is organized we shortly summarize its sections:

- Basic results on zippered rectangles are recalled in Section 2.1. We establish a compactness condition on the widths of the zippered rectangles whether they define the initial point of short complete paths in the Rauzy-Veech-diagram.
- Given two zippered rectangles, in Section 2.2 we explicitly give a quasi-conformal map between the two underlying Riemann surfaces.
- Section 2.3 reviews the action of an RV-step on the heights of zippered rectangles. The results are the basic ingredients to compute the behavior of the upper bound on Teichmüller distance along pairs of Teichmüller rays.
- Within Section 2.4 we relate the results from Section 2.3 to Teichmüller distance and see that Teichmüller distance decreases by a certain factor along every complete RV-path if the zippered rectangle fulfill some compactness condition on the heights data.
- We give examples to illustrate what may happen if zippered rectangles do not fulfill the compactness conditions. The examples suggest under which circumstances we can hope to find nice zippered rectangles. These examples are given in Section 2.5.
- Section 2.6 is devoted to show that zippered rectangles fulfill the compactness conditions as soon as the corresponding Teichmüller rays stay in the thick part of the stratum uninterrupted for a long time: For zippered rectangles not meeting the compactness conditions on the heights and widths we construct saddle connections which force the Teichmüller rays to leave the thick part of the stratum.
- All these results are put together in Section 2.7. This leads to a rate Teichmüller distance decreases with depending on the amount of time the

[10]This holds for an uncountable family of measures.

Teichmüller rays spend in the thick part of the stratum. Using Birkhoff's ergodic theorem we prove Theorem 2.1.

- In Section 2.8 we give an example of a Teichmüller ray illustrating why the Teichmüller rays have to spend time in the thick part uninterrupted before one can be sure that the Teichmüller distance has decreased.

2.1. **Interval exchange transformations and zippered rectangles.** The concept of zippered rectangles is important in our proof of Theorem 2.1. Zippered rectangles, due to Veech, are a way to describe flat surfaces using the theory of interval exchange maps. We introduce notations in this section and establish first results. For a more detailed treatment on these concepts we refer the reader to Viana [Via06] or Zorich [Zor06], for instance, and to Veech's original paper [Vee82].

Let $\mathcal{F}$ be an oriented minimal measured foliation on S, realized without saddle connections. Let I be any transverse interval whose left endpoint is a singularity of $\mathcal{F}$ and which avoids any singularity elsewhere. The first return map to I along leaves of $\mathcal{F}$ gives rise to an *interval exchange transformation* on I which we describe by a pair (π, w). Here $w = (w_j)_{j\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ is the vector of the lengths of the subintervals I_j, $j \in \mathcal{A}$, $\mathcal{A}$ a finite alphabet with $|\mathcal{A}| = n$ letters, and $\pi = (\pi^0, \pi^1)$ describes the *combinatorics*: the bijections $\pi^i : \mathcal{A} \to \{1, \dots, n\}$ give the ordering of the subintervals before and after the first return map is applied. The *permutation of an interval exchange transformation* is the permutation $\pi^1 \circ (\pi^0)^{-1}$ on the numbers $1, \dots, n$. By eventually shortening I we may and do assume that n is smallest possible. This minimal number depends on the equivalence class of $\mathcal{F}$ and is

$$|\mathcal{A}| = n = 2g - 1 + (\text{number of singularities of } \mathcal{F}) \geq 4.$$

Note that we assume $\mathcal{F}$ to be without saddle connections.

The *winner* of a Rauzy-Veech step is the longer one of the two subintervals $I_{(\pi^0)^{-1}(n)}$ (on the right of I before the first return map is applied) and $I_{(\pi^1)^{-1}(n)}$ (on the right of I after the first return map is applied). The *loser* is the shorter one. We use the label winner $=$ winner$^{(i)} \in \mathcal{A}$ to name the winner and the label loser $=$ loser$^{(i)} \in \mathcal{A}$ to name the loser after i Rauzy-Veech-steps. A *Rauzy-Veech-step* (RV-step for short) shortens the base interval from the right by the length of the loser. The new combinatorics are defined by the new first return map and the following rule: If the winner is $I_{(\pi^0)^{-1}(n)}$, the bijection π^0 stays unchanged, if the winner is $I_{(\pi^1)^{-1}(n)}$, the bijection π^1 stays unchanged. As a consequence, all lengths of subintervals remain unchanged but the length of the winner. Use upper indices on the lengths to indicate the number of RV-steps performed. Hence the action of an RV-step on the vector of lengths is just

$$w^{(i+1)}_{\text{winner}^{(i)}} = w^{(i)}_{\text{winner}^{(i)}} - w^{(i)}_{\text{loser}^{(i)}}.$$

As the bijection of the combinatorics which gives the winner-label stays unchanged, we have the following:

Corollary 2.2. *Let $(k, l) = (winner^{(i)}, loser^{(i)})$ be the pair of the winner- and the loser-label at the i-th RV-step and let $(k', l') = (winner^{(i+1)}, loser^{(i+1)})$ be the corresponding pair at the $(i+1)$-st RV-step. Then $k \in \{k', l'\}$.*

Recall that for any $X \in \mathscr{T}(S)$ the Hubbard-Masur Theorem 1.1 defines a quadratic differential $q = q(X, \mathcal{F})$ and hence a flat metric. Suppose that the vertical foliation of q is orientable and without saddle connections. For short intervals I

there is a homotopy along vertical leaves bringing I into horizontal position. This is always possible if the length of I is less than the length of the shortest saddle connection of q, for instance. Starting from a horizontal interval I and following the vertical leaves in the flat metric in upward direction leads to a *zippered rectangle* $\mathfrak{zr}(X, \mathcal{F}, I)$ based on the interval exchange map given by the first return map. The zippered rectangle covers the whole surface[11]. The rectangle of $\mathfrak{zr}(X, \mathcal{F}, I)$ which has I_j as lower horizontal side is denoted R_j, its height is denoted h_j and its width is exactly w_j, $j \in \mathcal{A}$. Let $h = (h_j)_{j\in\mathcal{A}}$ be the vector of the heights.

There is a piecewise isometric self-map $s : \mathcal{B} \to \mathcal{B}$ defined on the union $\mathcal{B}$ of the rectangle sides such that we get back to the flat metric by identifying boundary points $p \in \mathcal{B}$ and $s(p) \in \mathcal{B}$. Geometrically we glue the zippered rectangle sides by translations in a similar way as we did with polygons in Section 1.4.1.

Definition. The vertical sides of the rectangles are divided into subarcs which are maximal subsegments of the vertical sides with respect to the condition that the self-map s restricted to the segment is an isometry. We call these subarcs *glueing parts.*

For fixed combinatorics π, the lengths of the glueing parts depend linearly on the heights h. The triple (π, w, h) completely describes the zippered rectangle. On the other hand, given the zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$ we can not recover $(X, \mathcal{F})$ from (π, w, h) as the zippered rectangle does not give any information on the marking $f : S \to X$.

The action of an RV-step on $\mathfrak{zr}(X, \mathcal{F}, I)$ is as follows: The rightmost part of width w_l of the rightmost rectangle is cut off and put on top of some other rectangle. The combinatorics changes such that all heights stay unchanged but the height of the loser. We use upper indices on the heights to indicate the number of RV-steps performed. With this convention we have:

$$h^{(i+1)}_{\mathrm{loser}^{(i)}} = h^{(i)}_{\mathrm{loser}^{(i)}} + h^{(i)}_{\mathrm{winner}^{(i)}}.$$

This action can be expressed by matrix multiplication: $h^{(i+1)} = V^{(i)}h^{(i)}$, where

$$V^{(i)} = \mathbf{1} + E_{\mathrm{loser},\mathrm{winner}}$$

is the identity matrix with an additional 1 in position (loser, winner). Let

$$B^{(k)} = V^{(k)} \cdots V^{(1)}$$

be the product of the first k matrices, hence $h^{(k)} = B^{(k)}h^{(0)}$. Note that, if $I^{(i)}$ is horizontal, $I^{(i+1)}$ will still be horizontal.

Definition. For a given $\delta > 0$ we will say that I is of *admissible length* if the length of I is between δ and $\delta/2$, when realized horizontally.

Given X, $\mathcal{F}$ and I as above we can follow the Teichmüller geodesic $\rho_{X,\mathcal{F}}$. This results in increasing length of I. At some point the length of I will exceed δ. In order to get back to intervals of admissible length we apply an RV-step. The Teichmüller time between RV-step i and RV-step $i + 1$ depends only on the length of the loser in step i: after time $2\log(\delta/(\delta - w^{(i)}_{\mathrm{loser}}))$ we are back to intervals of length δ. Note that $w^{(i)}_{\mathrm{loser}} < \delta/2$.

[11]Interval exchange maps and zippered rectangles are defined for non-minimal foliations, too. In this case the cardinality of the alphabet $\mathcal{A}$ may be smaller, even equal to one, and the zippered rectangle maybe does not cover the whole surface.

An *RV-diagram* is an oriented graph defined in terms of interval exchange transformations and RV-steps. The vertices of an RV-diagram are the elements of a given equivalence class of permutations, where two permutations π_1 and π_2 belong to the same class if there exists an interval exchange transformation with permutation π_1 and a sequence of RV-steps leading to an interval exchange transformation with permutation π_2. There is an oriented edge from π_1 to π_2 if there is a single RV-step from π_1 to π_2. For every vertex there are two incoming and two outgoing edges, one for the case that the winner is on the top side of I and the second one for the case that the winner is on the bottom side. The choice of which side is the top side and which one the bottom side is given by the convention that we follow the leaves of the oriented foliation $\mathcal{F}$ in "upward" direction. Every interval exchange transformation and hence every zippered rectangle gives rise to an infinite sequence of RV-steps and thus defines a path in an RV-diagram. Let X, $\mathcal{F}$ and I be as above. The Teichmüller ray $\rho_{X,\mathcal{F}}$ projects to a path $\eta = \eta(X, \mathcal{F}, I)$ in the RV-diagram via $\mathfrak{zr}(X, \mathcal{F}, I)$. A finite subpath is *semi-complete* if every label is winner of at least one edge in that path. A finite subpath η^* is *complete* if the corresponding matrix B^{η^*} is positive, i.e. every entry of B^{η^*} is positive: $B^{\eta^*}_{(a,b)} > 0$ for all $a, b \in \mathcal{A}$.

The Teichmüller distance along pairs of Teichmüller rays with common uniquely ergodic vertical foliation will turn out to decrease every time a complete RV-subpath is covered. Thus we need a tool to locate Teichmüller geodesic segments which project to complete RV-paths.

Lemma 2.3. *Every finite RV-path η^* which is the concatenation of $2n-2$ semi-complete paths is a complete path, where $n = |\mathcal{A}|$.*

The proof of the lemma is based on the fact that the product of several non-negative matrices is a positive matrix if only the non-zero entries of the matrices are well distributed. This would not be the case if all matrices had the same block structure. First we exclude this case.

Proposition 2.4. *Suppose that η is a finite RV-path such that after the last step there is a label $k \in \mathcal{A}$ and a partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_0$ into two disjoint subsets with $B^{\eta}_{(l_1,k)} > 0$ for every $l_1 \in \mathcal{A}_1$ and $B^{\eta}_{(l_0,k)} = 0$ for every $l_0 \in \mathcal{A}_0$. Let η^* be the concatenation of η followed by two semi-complete paths. Then there is at least one $l^* \in \mathcal{A}_0$ with $B^{\eta^*}_{(l^*,k)} > 0$.*

Proof. Within the first semi-complete path following η there will be an RV-step with a winner-label in $\mathcal{A}_0$. Let it be the a-th step. Within the second semi-complete path there will be an RV-step with winner-label in $\mathcal{A}_1$. Let $b > a$ be the smallest integer larger than a with winner-label in $\mathcal{A}_1$, thus $\mathrm{winner}^{(b)} \in \mathcal{A}_1$ and $\mathrm{winner}^{(b-1)} \in \mathcal{A}_0$. The b-th RV-step is in the first or in the second semi-complete path following η. As $\mathcal{A}_1$ and $\mathcal{A}_0$ are disjoint, Corollary 2.2 implies $\mathrm{loser}^{(b)} = \mathrm{winner}^{(b-1)}$. Let l^* be this label, hence $V^{(b)} = \mathbf{1} + E_{l^*,\mathrm{winner}^{(b)}}$. By assumption $B^{\eta}_{(\mathrm{winner}^{(b)},k)}$ is positive, thus $B^{\eta^*}_{(l^*,k)} > 0$. □

Lemma 2.3 is an immediate consequence of the proposition.

Proof of Lemma 2.3. Let $k \in \mathcal{A}$ be arbitrary but fixed. After each RV-step partition the alphabet $\mathcal{A}$ into two disjoint subsets $\mathcal{A} \in \mathcal{A}_1 \sqcup \mathcal{A}_0$ as in Proposition 2.4 (in our notation we suppress the dependence on k). After two semi-complete paths the cardinality of $\mathcal{A}_0$ decreases by at least one. As there are only $n = |\mathcal{A}|$ labels in the alphabet, after $n-1$ iterations $\mathcal{A}_0$ is void. $\square$

We are ready to prove that semi-complete (and hence complete) paths will show up after a uniformly bounded number of RV-steps if the widths are not too small.

Proposition 2.5. *Let X, $\mathcal{F}$ and I be such that $\mathfrak{zr}(X, \mathcal{F}, I)$ is a zippered rectangle as above. Suppose that there is a compact interval $\mathcal{I} = [\alpha, \beta] \subset (0, \delta)$ such that all widths of rectangles are within $\mathcal{I}$ along the Teichmüller ray $\rho_{X,\mathcal{F}}$ (clearly, $\alpha < \delta/2$). After a finite number of RV-steps, the subpath η^* in $\eta(X, \mathcal{F}, I)$ will be complete. Moreover, the length of the preimage of η^* on the Teichmüller ray $\rho_{X,\mathcal{F}}$ is bounded from above and from below.*

Proof. Obviously each width in the zippered rectangle has to be shortened along the Teichmüller ray after a time span of length at most $2\log(\beta/\alpha)$. Widths are shortened only when the corresponding subinterval is a winner. Hence after a time span of length $2\log(\beta/\alpha)$ we see a semi-complete path in the RV-diagram. Lemma 2.3 gives the statement on the length-bounds for the shortest initial subsegment of $\rho_{X,\mathcal{F}}$ projecting to a complete path. In the discussion above we saw that there is a time-span of at least $2\log(\delta/(\delta-\alpha))$ and at most $2\log(2)$ between two RV-steps, which gives the bound on the number of RV-steps. $\square$

Remark. There are two important remarks on this proposition. Let the conditions of Proposition 2.5 be fulfilled.

The upper bound $T_c > 0$ for the length of the initial segment of $\rho_{X,\mathcal{F}}$ which projects to a complete RV-subpath is independent of the measured foliation. For fixed $\delta > 0$ it depends only on the compact interval $\mathcal{I}$ and on the number n of subintervals.

Let $T_c > 0$ be this uniform upper bound. If we imply that for a time-span of at least T_c the widths of the zippered rectangle are within $\mathcal{I}$, the proposition holds.

Thus we can restate the proposition under weaker assumptions:

Corollary 2.6. *For $\delta > 0$, $0 < \alpha < \delta/2$, $\alpha < \beta < \delta$ and $n \in \mathbb{N}$ there exists $T_c = T_c(\alpha, \beta, \delta, n) > 0$ with the following property:*
Let X, $\mathcal{F}$ and I be such that $\mathfrak{zr}(X, \mathcal{F}, I)$ is a zippered rectangle as in Proposition 2.5 and with n rectangles. Suppose that all widths of rectangles are within the compact interval $[\alpha, \beta] \subset (0, \delta)$ along the initial segment of length T_c of the Teichmüller ray $\rho_{X,\mathcal{F}}$. This segment projects to a complete path of uniformly bounded length in the RV-diagram.

2.2. **Zippered rectangles and pairs of Teichmüller geodesics.** Given two zippered rectangles one might want to compare them and extract information on the distance between the underlying Riemann surfaces. For the pairs of zippered rectangles we will deal with, this turns out to be possible.

Let $\mathcal{F}$ be an oriented measured foliation on S and let X and $\tilde{X}$ be two points in Teichmüller space. Let I be an interval transverse to $\mathcal{F}$ and subject to the condition that it can be realized horizontally on $(X, \mathcal{F})$, starting at a singularity and avoiding any singularity elsewhere, and of admissible length for some fixed $\delta > 0$. Suppose that $(X, \mathcal{F})$ and $(\tilde{X}, \mathcal{F})$ belong to the same stratum and that the Teichmüller distance between X and $\tilde{X}$ is small enough. The map $q(\,\cdot\,, \mathcal{F})$ in the Hubbard-Masur Theorem 1.1 is continuous, which implies that I can be realized simultaneously as horizontal intervals on $(X, \mathcal{F})$ and $(\tilde{X}, \mathcal{F})$. Hence the induced interval exchange transformations and RV-paths are the same if $\mathcal{F}$ is minimal and oriented and if the realizations on X and $\tilde{X}$ are without vertical saddle connections. We emphasize that X and $\tilde{X}$ have to be of small distance. The only datum that differs for the zippered rectangles $\mathfrak{zr}(X, \mathcal{F}, I)$ and $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I)$ is the heights datum, and this datum can easily be compared. Thus we make the following definition.

Definition. Let $\mathcal{F}$ be a minimal measured foliation (not necessarily orientable) and let X and $\tilde{X}$ be two points in Teichmüller space. Suppose that both quadratic differentials $(X, \mathcal{F})$ and $(\tilde{X}, \mathcal{F})$ are without vertical saddle connections and belong to the same stratum. The pair of Teichmüller rays determined by $(X, \mathcal{F})$ and $(\tilde{X}, \mathcal{F})$ is called a *knorke pair*[12] in direction $\mathcal{F}$ if both quadratic differentials are of unit norm (note that we assume the vertical measured foliations are exactly the same, not only being in the same projective class). If there is no confusion to expect which measured foliation is meant we call $(X, \tilde{X})$ a knorke pair. Each of the two entries of the pair is called a *knorke partner*. We use the terms knorke pair and knorke partner not only for the Teichmüller rays but for the defining quadratic differentials, too.

Remark. Suppose that $\mathcal{F}$ is an uniquely ergodic measured foliation and let $(X, \tilde{X})$ be a knorke pair. Masur proved that both rays have the same endpoint in the Thurston boundary, [Mas82b].

Let a knorke pair $(X, \tilde{X})$ in direction of an oriented minimal measured foliation $\mathcal{F}$ be given. Let I be a transversal interval of admissible length, such that the realizations on $(X, \mathcal{F})$ and $(\tilde{X}, \mathcal{F})$ are horizontal, start at a singularity and avoid any singularity elsewhere. Thus we get two zippered rectangles $\mathfrak{zr}(X, \mathcal{F}, I)$ and $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I)$. For the heights, widths and so on which belong to the zippered rectangle $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I)$, we use the old symbols, but with a tilde on top, e.g. $\tilde{h} = (\tilde{h}_j)_{j \in \mathcal{A}}$ is the vector of the heights of the rectangles. In his 1980 paper [Mas80] Masur computed the quasi-conformality constant of a quasi-conformal map defined in terms of the zippered rectangles. This homeomorphism is a piecewise affine stretch along vertical lines of the zippered rectangles; the interior of R_j is mapped homeomorphically onto the interior of $\tilde{R}_j$, $j \in \mathcal{A}$, singularities are mapped to singularities, glueing parts of vertical sides are mapped linearly to glueing parts of vertical sides, the vertical line in R_j dividing R_j into two parts of equal size is mapped linearly to the vertical line in $\tilde{R}_j$ with the same property, and the map is the identity on I. The restriction of this map to one rectangle is sketched in Figure 1. Let the coordinates of the shaded parts be $(0,0)$, $(a,0)$, (a,c) and $(0,b)$ for the rectangle R and $(0,0)$, $(\tilde{a},0)$, $(\tilde{a},\tilde{c})$ and $(0,\tilde{b})$ for the rectangle $\tilde{R}$. The shaded part on $\tilde{R}$ is the image of the shaded part on R. Note that $a = \tilde{a} = w_R/2$ and $b = c$, but not in general $\tilde{b} = \tilde{c}$. We have $\tilde{b}/b = \tilde{h}_R/h_R$. The partial derivatives of Masur's quasi-conformal map can explicitly be computed, see [Mas80]. As an example we give the partial derivatives for the restriction of the map to the shaded part in Figure 1. Let R be given in the

[12]'Knorke' is a nowadays very seldom used old German expression meaning great, awesome, fabulous. In the author's ears this word sounds knorke.

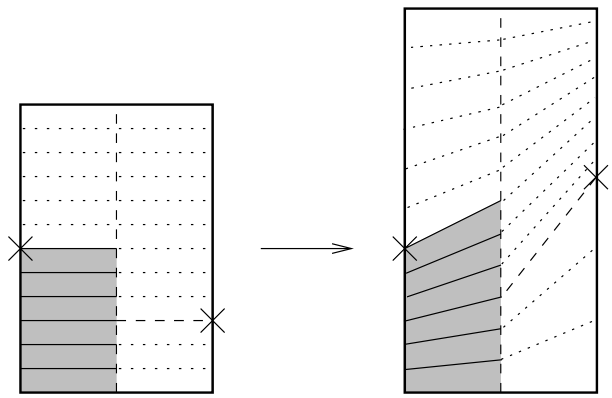

FIGURE 1. The restriction of Masur's quasi-conformal map to one rectangle

(x, y)-plane and $\tilde{R}$ in the (u, v)-plane. We get:

$$u_x = 1 \qquad v_x = \frac{y}{a}\left(\frac{\tilde{c}}{c} - \frac{\tilde{b}}{b}\right)$$

$$u_y = 0 \qquad v_y = \frac{x}{a}\left(\frac{\tilde{c}}{c} - \frac{\tilde{b}}{b}\right) + \frac{\tilde{b}}{b}.$$

Thus the (restriction of the) map is $\frac{Q}{2} + \sqrt{\frac{Q^2}{4} - 1} = K$-quasi-conformal, where

$$\begin{aligned} Q &= \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{u_x v_y - u_y v_x} = v_y + \frac{1 + v_x^2}{v_y} \\ &= \left(\frac{x}{a}\left(\frac{\tilde{c}}{c} - \frac{\tilde{b}}{b}\right) + \frac{\tilde{b}}{b}\right) + \frac{1 + \frac{y^2}{a^2}\left(\frac{\tilde{c}}{c} - \frac{\tilde{b}}{b}\right)^2}{\frac{x}{a}\left(\frac{\tilde{c}}{c} - \frac{\tilde{b}}{b}\right) + \frac{\tilde{b}}{b}} \\ &\geq 2, \end{aligned} \tag{1}$$

and hence K is bounded from above by $g(w) \cdot M$, where g depends continuously on the widths and M is the maximum of all quotients of the form length of glueing part p in $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I)$ by length of glueing part p in $\mathfrak{zr}(X, \mathcal{F}, I)$ or length of side s in $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I)$ by length of side s in $\mathfrak{zr}(X, \mathcal{F}, I)$, and their inverses.

Corollary 2.7. *If we apply the Teichmüller flow* $\operatorname{diag}(\lambda, \lambda^{-1})$, $\lambda \to \infty$, *without applying RV-steps, the value Q given by Formula* (1) *will tend to $v_y + 1/v_y$ from above: $Q \searrow v_y + 1/v_y > 2$. Hence the quasi-conformality constant is decreasing, but it is bounded from below by a number strictly bigger than* 1.

Recall that the lengths of the glueing parts depend linearly on the heights. Suppose that the widths are within a fixed compact interval. Then

$$D_{\mathcal{F},I}(X, \tilde{X}) = \max_{1 \leq j \leq n} \left\{\log(h_j/\tilde{h}_j), \log(\tilde{h}_j/h_j)\right\}$$

bounds Teichmüller distance up to some uniform factor: $d_{\mathscr{T}}(X, \tilde{X}) \dot{\prec} D_{\mathcal{F},I}(X, \tilde{X})$. (Recall the definition of the Teichmüller distance as in Section 1.1.3.)

In the above setting the only difference between $\mathfrak{zr}(X,\mathcal{F},I)$ and $\mathfrak{zr}(\tilde{X},\mathcal{F},I)$ is in the heights. Let

$$\tau = \tilde{h} - h$$

be the vector of difference in heights of the two zippered rectangles $\mathfrak{zr}(X,\mathcal{F},I)$ and $\mathfrak{zr}(\tilde{X},\mathcal{F},I)$. The property of a zippered rectangle $\mathfrak{zr}(X,\mathcal{F},I)$ having unit area (which is equivalent to the quadratic differential having unit area) can be expressed via the Euclidean scalar product:

$$1 = \operatorname{area}(\mathfrak{zr}(X,\mathcal{F},I)) = \langle w, h\rangle.$$

Suppose both quadratic differentials $(X,\mathcal{F})$ and $(\tilde{X},\mathcal{F})$ have unit area. Using the above formula we get

$$0 = \langle w, \tilde{h}\rangle - \langle w, h\rangle = \langle w, \tau\rangle,$$

and see that for $\tau \neq 0$ at least one entry of τ has to be positive and at least one entry of τ has to be negative. Our next step will be to analyse the behavior of $D_{\mathcal{F},I}$ along knorke rays. The vector τ will simplify this analysis.

2.3. **Rauzy-Veech steps and the magnitude of height differences.** Let the pair $(X,\tilde{X})$ be a knorke pair in direction of an orientable minimal measured foliation $\mathcal{F}$ and let I be transversal and of admissible length for both knorke partners. We will examine how the upper bound $D_{\mathcal{F},I}(X,\tilde{X})$ on Teichmüller distance behaves under RV-steps. As in the previous section, let h and $\tilde{h}$ be the height vectors. Let $q_j^{(i)} = \tilde{h}_j^{(i)}/h_j^{(i)}$ be the quotient of the heights of the rectangles labeled with j after the i-th RV-step. As mentioned in Section 2.1, only the loser's value changes in each RV-step. We compute

$$\begin{aligned} q^{(i+1)}_{\mathrm{loser}^{(i)}} &= \frac{\tilde{h}^{(i+1)}_{\mathrm{loser}^{(i)}}}{h^{(i+1)}_{\mathrm{loser}^{(i)}}} = \frac{\tilde{h}^{(i)}_{\mathrm{loser}^{(i)}} + \tilde{h}^{(i)}_{\mathrm{winner}^{(i)}}}{h^{(i)}_{\mathrm{loser}^{(i)}} + h^{(i)}_{\mathrm{winner}^{(i)}}} \\ &= \frac{h^{(i)}_{\mathrm{loser}^{(i)}}}{h^{(i)}_{\mathrm{loser}^{(i)}} + h^{(i)}_{\mathrm{winner}^{(i)}}} \frac{\tilde{h}^{(i)}_{\mathrm{loser}^{(i)}}}{h^{(i)}_{\mathrm{loser}^{(i)}}} + \frac{h^{(i)}_{\mathrm{winner}^{(i)}}}{h^{(i)}_{\mathrm{loser}^{(i)}} + h^{(i)}_{\mathrm{winner}^{(i)}}} \frac{\tilde{h}^{(i)}_{\mathrm{winner}^{(i)}}}{h^{(i)}_{\mathrm{winner}^{(i)}}} \\ &= \mu^{(i)} q^{(i)}_{\mathrm{loser}^{(i)}} + (1-\mu^{(i)}) q^{(i)}_{\mathrm{winner}^{(i)}} \end{aligned}$$

Remark. Note that $\mu^{(i)} = h^{(i)}_{\mathrm{loser}^{(i)}}/(h^{(i)}_{\mathrm{loser}^{(i)}} + h^{(i)}_{\mathrm{winner}^{(i)}}) \in (0,1)$ depends continuously on the data $h^{(i)}$ and does not depend on $\tilde{h}^{(i)}$. The new $q_j^{(i+1)}$ are convex combinations of the old quotients. Moreover the new quotient $q^{(i+1)}_{\mathrm{loser}^{(i)}}$ of the loser is a strict convex combination. As a consequence, RV-steps will decrease the maximum of the height quotients. The quotients do not change under Teichmüller flow as long as no RV-step occurs.

If we look at what happens to the quotients $\tau_j/h_j = q_j - 1$ instead of the quotients $q_j = \tilde{h}_j/h_j$, we will see exactly the same convex combinations:

$$\frac{\tau^{(i+1)}_{\mathrm{loser}^{(i)}}}{h^{(i+1)}_{\mathrm{loser}^{(i)}}} = \mu^{(i)} \frac{\tau^{(i)}_{\mathrm{loser}^{(i)}}}{h^{(i)}_{\mathrm{loser}^{(i)}}} + (1-\mu^{(i)}) \frac{\tau^{(i)}_{\mathrm{winner}^{(i)}}}{h^{(i)}_{\mathrm{winner}^{(i)}}}.$$

For us these quotients will have the advantage that we just have to keep track of the data given by $\mathfrak{zr}(X,\mathcal{F},I)$ and some additional vector τ; we can forget about $\tilde{X}$.

In this sense one should think of τ being some kind of tangential vector to zippered rectangles.

Definition. The *magnitude* of the vector τ based at $\mathfrak{zr}(X,\mathcal{F},I)$ is defined as

$$\|\tau\|^* = \|\tau\|^*_{X,\mathcal{F},I} = \max_{j\in\mathcal{A}} \frac{|\tau_j|}{h_j}$$

Remark. The magnitude is a good estimate for Teichmüller distance. Let X and $\tilde{X}$ be two points of small distance in Teichmüller space and let τ be the height difference between the zippered rectangles $\mathfrak{zr}(X,\mathcal{F},I)$ and $\mathfrak{zr}(\tilde{X},\mathcal{F},I)$. Under some compactness conditions on the zippered rectangles, the magnitude and Teichmüller distance have bounded quotients: $\|\tau\|^* \stackrel{\cdot}{\asymp} d_{\mathscr{T}}(X,\tilde{X})$, c.f. Lemma 2.13.

In fact, the magnitude can be seen as a norm function $\mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Indeed, in the language of linear algebra we have $\|\tau\|^*_{X,\mathcal{F},I} = |\mathrm{diag}(h_1,\cdots,h_n)^{-1}\cdot\tau|_{\max}$. Even if a vector $\tau' \in \mathbb{R}^n$ does not arise as a height difference, we can compute its magnitude $\|\tau'\|^*_{X,\mathcal{F},I}$. In particular, consider the following example. Let τ be a vector of height differences as above. It may happen that the vector $\tau' = -\tau$ does not lead to a zippered rectangle with heights $h_j - \tau_j$ as these values may be non-positive. But still one can compute the magnitude and get $\|-\tau\|^*_{X,\mathcal{F},I} = \|\tau\|^*_{X,\mathcal{F},I}$. In this spirit, for a vector τ defined as the height difference between two zippered rectangles $\mathfrak{zr}(X,\mathcal{F},I)$ and $\mathfrak{zr}(\tilde{X},\mathcal{F},I)$, both magnitudes $\|\tau\|^*_{X,\mathcal{F},I}$ and $\|\tau\|^*_{\tilde{X},\mathcal{F},I}$ are meaningful.

Lemma 2.8. *The magnitude $\|\tau^{(i)}\|^*$ decreases along the RV-path $\eta(X,\mathcal{F},I)$ determined by X, $\mathcal{F}$ and I. Moreover, the factor by which the magnitude is scaled along a finite subpath depends continuously on the heights h of the zippered rectangle $\mathfrak{zr}(X,\mathcal{F},I)$ and the vector τ, and depends on the finite path in the RV-diagram. If the subpath is complete, the factor will be strictly less than* 1.

Proof. All but the last statement follow immediately from the definition of τ and the behavior of the q_j. To see the last statement, let $A \in \mathcal{A}$ be the index giving the maximum in $\|\tau^{start}\|^*$ and let $B \neq A$ be such that $\tau_A^{start}\cdot\tau_B^{start} < 0$. At least one label $B \in \mathcal{A}$ has this property: w is a positive vector and $\langle w,\tau\rangle = 0$. As the path is complete, the (A,B)-entry of the RV-matrix is positive, thus after following the complete RV-path, τ_A^{end}/h_A^{end} will be a strict convex combination involving $\tau_A^{start}/h_A^{start}$ and $\tau_B^{start}/h_B^{start}$. □

The lemma implies that, given a complete RV-path of bounded length, there is an uniform upper bound strictly less than one for the scaling factor of the magnitudes if, first, the heights of the zippered rectangles are bounded from below by a number greater zero and bounded from above and, second, the entries of the vectors $\tau^{(i)}$ are within a compact interval. We will find conditions under which the zippered rectangles along a knorke pair are in such a setting.

2.4. **The magnitudes and Teichmüller distance.** Let $(X,\tilde{X})$ be a knorke pair in direction of an orientable minimal measured foliation $\mathcal{F}$ and let I be a transversal interval which is of admissible length for both knorke partners. We just established that under certain conditions we can control how fast the magnitudes decrease. Our aim is to control the Teichmüller distance along knorke pairs. Thus we want to relate Teichmüller distance to the magnitude. To this end we would like to bound $D_{\mathcal{F},I}$ from above by the magnitude and apply the results of Section 2.2.

Unfortunately the magnitude is not symmetric in X and $\tilde{X}$, whereas $D_{\mathcal{F},I}(X,\tilde{X})$ is symmetric, so we should not hope that the magnitude bounds this value in general. If we constrain to height differences with small magnitudes we can control the error. This result will be strong enough to give a bound on the factor by which Teichmüller distance decreases along complete paths.

As a first step, we define an upper bound for $D_{\mathcal{F},I}(X,\tilde{X})$ in terms of τ. Later we show that under certain conditions this bound is roughly $\|\tau\|^*_{X,\mathcal{F},I}$.

Definition. For any pair of zippered rectangles $\mathfrak{zr}(X,\mathcal{F},I)$ and $\mathfrak{zr}(\tilde{X},\mathcal{F},I)$ with height difference $\tau = \tilde{h} - h$ define $n(X,\tilde{X},\mathcal{F},I) = \max\bigl\{\|\tau\|^*_{X,\mathcal{F},I}, \|\tau\|^*_{\tilde{X},\mathcal{F},I}\bigr\}$

Note that we can express this value completely in terms of h and τ:

$$n(X,\tilde{X},\mathcal{F},I) = \max_{1\le j\le n}\Bigl\{\frac{|\tau_j|}{h_j}, \frac{|\tau_j|}{h_j+\tau_j}\Bigr\}.$$

Lemma 2.9. *This value is an upper bound for Teichmüller distance:*

$$d_{\mathscr{T}}(X,\tilde{X}) \dot{\prec} \log\bigl(1 + n(X,\tilde{X},\mathcal{F},I)\bigr) \le n(X,\tilde{X},\mathcal{F},I),$$

if the zippered rectangles $\mathfrak{zr}(X,\mathcal{F},I)$ and $\mathfrak{zr}(\tilde{X},\mathcal{F},I)$ have uniformly bounded withs.

Proof. The first inequality is due to Masur [Mas80] (compare Section 2.2), as the value $D_{\mathcal{F},I}(X,\tilde{X}) = \max_j\bigl\{\log\bigl((h_j+\tau_j)/h_j\bigr), \log\bigl(h_j/(h_j+\tau_j)\bigr)\bigr\}$ is bounded from above by $\log(1 + n(X,\tilde{X},\mathcal{F},I))$: Suppose that the maximum in the definition of $D_{\mathcal{F},I}(X,\tilde{X})$ is achieved by the index j. If $\tau_j > 0$, we are done by the following:

$$D_{\mathcal{F},I}(X,\tilde{X}) = \log\Bigl(\frac{h_j+\tau_j}{h_j}\Bigr) = \log\Bigl(1 + \frac{\tau_j}{h_j}\Bigr) = \log\Bigl(1 + \frac{|\tau_j|}{h_j}\Bigr).$$

Suppose $\tau_j < 0$. Then

$$D_{\mathcal{F},I}(X,\tilde{X}) = \log\Bigl(\frac{h_j}{h_j+\tau_j}\Bigr) = \log\Bigl(1 - \frac{\tau_j}{h_j+\tau_j}\Bigr) = \log\Bigl(1 + \frac{|\tau_j|}{h_j+\tau_j}\Bigr),$$

hence we are done again.
The second inequality comes from the fact that $\frac{\mathrm{d}}{\mathrm{d}y}\log(1+y) \le 1 = \frac{\mathrm{d}}{\mathrm{d}y}y$, $y > 0$, and that $\log 1 = 0$. □

The next proposition shows that for height differences with small magnitudes, the values of n and $\|\cdot\|^*$ are the same up to some uniform multiplicative error.

Proposition 2.10. *Fix $0 < c < 1/2$ small. Suppose that the height difference τ is small, too: $\|\tau\|^*_{X,\mathcal{F},I} < c$. Then $\|\tau\|^*_{X,\mathcal{F},I} \dot{\asymp} n(X,\tilde{X},\mathcal{F},I)$.*

Proof. For $\|\tau\|^*_{X,\mathcal{F},I} < c$ it holds $\|\tau\|^*_{X,\mathcal{F},I}/\|\tau\|^*_{\tilde{X},\mathcal{F},I} \in (1-c, 1+c)$ as

$$\frac{|\tau_i|/h_i}{|\tau_i|/(h_i+\tau_i)} = \frac{h_i+\tau_i}{h_i} = 1 + \frac{\tau_i}{h_i}.$$

□

Corollary 2.11. *For zippered rectangles as above and such that the height difference has small magnitudes, the magnitude bounds Teichmüller distance. To be exact, let $(X,\tilde{X})$ be a knorke pair in direction $\mathcal{F}$, let I be transversal of admissible length on both knorke partners and let $\tau = \tilde{h} - h$ have a magnitude small enough to apply Proposition 2.10. Suppose that the widths of the subintervals are within a fixed compact interval. Then $d_{\mathscr{T}}(X,\tilde{X}) \dot{\prec} \|\tau\|^*_{X,\mathcal{F},I}$.*

Now we can prove a lemma that tells us under which condition we can expect that the Teichmüller distance will decrease exponentially. An informal way to phrase the lemma is: If all zippered rectangle data are bounded for a long time, the knorke partners will get closer. The formal statement is in

Lemma 2.12. *Let $(X, \tilde{X})$ be a knorke pair in direction $\mathcal{F}$, I a transversal interval of admissible length on both knorke partners, and let $\tau = \tilde{h} - h$ have a magnitude small enough to apply Proposition 2.10. Furthermore let $\mathcal{I} = [\alpha, \beta] \subset (0, \delta)$ and $\mathcal{I}^* = [A, B] \subset \mathbb{R}_{>0}$ be compact. Let $T_c = T_c(\alpha, \beta, \delta, n) > 0$ be as in Corollary 2.6, where n is determined by the measured foliation $\mathcal{F}$. Suppose that there is a set $K \subset \mathbb{N}$ such that the widths of the rectangles defined by $\mathfrak{zr}(X, \mathcal{F}, I)$ via RV-steps along the Teichmüller ray $\rho_{X,\mathcal{F}}$ are within $\mathcal{I}$ along disjoint segments $\rho^{(k)}$, $k \in K$, of length T_c on the Teichmüller ray $\rho_{X,\mathcal{F}}$, and that the heights are within $\mathcal{I}^*$ along the same Teichmüller segments.*
Then there exists a number $0 < \chi < 1$, depending on the RV-diagram defined by the zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$, on $\mathcal{I}$ and on $\mathcal{I}^*$, such that the magnitude of τ decreases at least by a factor of χ along each Teichmüller segment $\rho^{(k)}$. Hence the Teichmüller distance between $\rho_{X,\mathcal{F}}(t)$ and $\rho_{\tilde{X},\mathcal{F}}(t)$ decreases exponentially in the number of segments $\rho^{(k)}$ which are contained in the initial Teichmüller segment $\rho_{X,\mathcal{F}}([0, t])$.

Proof. By Corollary 2.6, each segment $\rho^{(k)}$, $k \in K$, projects to a complete RV-path. Lemma 2.8 implies that the magnitude decreases strictly, where the factor depends continuously on the heights and on τ if we fix the complete RV-path. As the heights are restricted to the compact interval $\mathcal{I}^*$, there is an upper bound $0 < \chi^{(k)} < 1$ on the factor, depending on the RV-path. The lengths of the possible complete RV-paths are bounded as of Corollary 2.6, hence there are only finitely many RV-paths which can occur. Thus the upper bound $\chi^{(k)} \leq \chi$ on the factor can be chosen to be uniform, and the magnitude decreases exponentially in the number of times we follow one of these complete RV-paths. Corollary 2.11 gives the statement on the Teichmüller distance along the segments $\rho^{(k)}$ on which the zippered rectangles data are within the compact intervals. For the times in between, we note that, if we do not apply RV-steps, the quasi-conformality constant of the piecewise affine stretch defined by Masur is non-increasing along Teichmüller geodesics, c.f. Corollary 2.7. $\square$

Remark. There are only finitely many possible RV-diagrams for each stratum. Hence the value χ can be chosen to depend on the stratum and the intervals $\mathcal{I}$ and $\mathcal{I}^*$ only.

We established an upper bound for Teichmüller distance. The remaining part of this section is to show that the magnitude serves – up to uniform factor – as a lower bound for Teichmüller distance as well, provided the zippered rectangles have bounded data. The argument is a simple compactness argument and relies on Veech's holonomy coordinates for orientable quadratic differentials. Under the additional assumption (which we did not need for our proof) that the heights of the zippered rectangles are within a compact interval, one can prove Corollary 2.11 with this argument, too.

In his paper [Vee90], Veech constructed local complex coordinates for orientable quadratic differentials with zeros of fixed orders and established that the orientable unit area quadratic differentials form a real analytic embedded submanifold. In fact, Veech's construction is carried out over Moduli Space, and it can be lifted to Teichmüller space. These coordinates are by now known as *holonomy coordinates* and can be interpreted as a set of complex numbers given by integrating the quadratic differential along a set of saddle connections which jointly cut the surface into a union of topological discs. Veech originally constructed the holonomy coordinates in terms of special rectangle decompositions of the flat metric (called a *weaver* in his terminology). Zippered rectangles can be seen as weavers, where the so-called *ends* are the base interval I and unions of at most two glueing parts on vertical sides. To give an example, in Figure 2 (Section 2.5) the weaver consists of the bold polygon and the three vertical dotted lines. Via its ends, a weaver defines a rectangular decomposition of the surface. Call maximal regular subsegments of sides of rectangles in this decomposition *edges* of the weaver. Let λ be the vector whose entries are the lengths of the edges. There exists an $\mathbb{R}$-linear map $(w, h) \mapsto \lambda$. This map has maximal rank. As the zippered rectangles which we consider have minimal vertical foliation without saddle connections, the real dimension of the stratum equals two times the number of subintervals of the underlying interval exchange transformation: $\dim_{\mathbb{R}}(\mathcal{Q}) = 2n$, where $n = |\mathcal{A}|$. Thus the map $(w, h) \mapsto \lambda$ is a linear isomorphism, which implies that the widths and heights of zippered rectangles are real analytic local coordinates for the corresponding stratum. In particular, for every $0 \neq \tau \in \mathbb{R}^n$ with $\langle w, \tau \rangle = 0$, the deformation $t \mapsto (w, h + t\tau)$ defines a real analytic path in the stratum with derivative of rank one locally at the origin. For every zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$, the magnitude $\|\tau\|^*_{X,\mathcal{F},I}$ defines a norm on $\mathbb{R}^n$. For a fixed τ, this norm only depends on the heights vector, and moreover, it depends continuously on the heights vector. Recall that the unit sphere in $\mathbb{R}^n$ is compact. Thus for every fixed $(w, h) \in \mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$ there exists a constant $c(w, h) > 0$ such that for $\mathfrak{zr}(X, \mathcal{F}, I) = (\pi, w, h)$ and $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I) = (\pi, w, h+\tau)$ with $\|\tau\|^*_{X,\mathcal{F},I} \leq 1/2$ and $\langle w, \tau \rangle = 0$ we have $d_{\mathscr{T}}(X, \tilde{X}) \geq c(w, h)\|\tau\|^*_{X,\mathcal{F},I}$, where $c(w, h)$ depends continuously on (w, h). Therefore, together with Corollary 2.11, we just established

Lemma 2.13. *Let $K \subset \mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$ be compact. Suppose $\mathfrak{zr}(X, \mathcal{F}, I) = (\pi, w, h)$ and $\mathfrak{zr}(\tilde{X}, \mathcal{F}, I) = (\pi, w, h + \tau)$ are unit area zippered rectangles, where $(w, h) \in K$ and $\|\tau\|^*_{X,\mathcal{F},I}$ is small enough to apply Proposition 2.10. Then $d_{\mathscr{T}}(X, \tilde{X}) \stackrel{.}{\asymp} \|\tau\|^*_{X,\mathcal{F},I}$.*

2.5. **Compact sets of zippered rectangles.** Up to now we established that the distance along a knorke pair decreases by a uniform factor if the heights and widths of the corresponding zippered rectangles are within compact intervals for a certain amount of time. This leads immediately to exponential asymptotics for knorke pairs whose zippered rectangles along the Teichmüller geodesics have data inside these compact sets. From a Teichmüller theoretic point of view this is not satisfactory: A Teichmüller ray does not come with a priori information on the zippered rectangles along the ray. We would like to compute the distance along knorke pairs in terms of the amount of time which the two rays spent in the thick part of Teichmüller space or in the thick part of the stratum. Moreover, at a first glance it seems reasonable to hope for a direct relation between the thick part of a stratum and a compact set of zippered rectangle data. Let the permutation π be fixed. The map $(\pi, w, h) \mapsto q(\pi, w, h)$ which maps a zippered rectangle to the underlying quadratic differential by glueing sides is locally continuous in w and h.

(Note however that this map is not well defined. We have to fix a marking and base interval for a zippered rectangles (π, w, h) and then we can vary the continuous data slightly. Doing so we get a locally defined continuous map.) Hence the image of a compact set of zippered rectangle data is a compact set in the thick part of the corresponding stratum. But the naive approach to use a classical continuity and compactness argument to show compactness of the space of zippered rectangles over the thick part of a given stratum fails.

To illustrate this fact we give a short sketch of how to see this. Fix an orientable quadratic differential $q \in \mathcal{Q}_\varepsilon$. Let p be our favourite singularity – this is a finite choice – with cone angle $l\pi$. For $\delta < \varepsilon/2$ every interval (geodesic segment) of length δ and starting at p misses every singularity elsewhere and hence gives rise to a zippered rectangle defined via the directional foliation orthogonal to the interval. Thus the set of zippered rectangles on q based on intervals of length δ starting at p can be parametrized by the "direction" $\theta \in [0, l\pi]/_{0\sim l\pi}$ of the interval. We denote the interval in direction θ and of length δ by I_θ. It is possible that the direction perpendicular to θ has a cylinder decomposition into cylinders of circumference larger than δ. Choose a direction θ with this property. The zippered rectangle based then on I_θ does not cover all of q, it just covers some part of the cylinder I_θ points into, and it consists of just one rectangle with area strictly less than one. But as minimal directions are dense in $\mathbb{S}^1$, for arbitrarily small changes of direction ζ we can find directions $\theta^* = \theta + \zeta$ with minimal perpendicular foliation. The corresponding zippered rectangles cover all of q, hence have area one. The area of a zippered rectangle can be expressed continuously in terms of the widths and heights (compare the last paragraph of Section 2.2). This shows the existence of underlying quadratic differentials q and lengths δ of the base interval with the property that the widths and heights of zippered rectangles do not vary continuously in the direction θ of the base interval I_θ.

Example 1: Degeneration of a subinterval

A question to ask is: What does actually happen in the stratum if we let a zippered rectangle degenerate, say, if we let the width of one subinterval tend to zero? The answer is somewhat surprising: There are examples where nothing special happens inside the stratum.

The object that changes is the vertical foliation. A degeneration of a subinterval may create a vertical saddle connection while all the quadratic differentials defined by the zippered rectangles still are points in the thick part of the stratum. We give an example.

Let $(\pi, w(l), h)_{l\in\mathbb{N}}$ be a sequence of zippered rectangles as in Figure 2, where π and h as well as $w(l)_j$, $j \in \{A, B\}$, and $w(l)_C + w(l)_D$ are constant along the sequence, and $w(l)_C \to 0$ for $l \to \infty$. Suppose that the widths of the zippered rectangles are such that the vertical foliations defined by the zippered rectangles are without saddle connections. (We will discuss the outcome of different choices of widths in the next paragraph.) The area of the zippered rectangle decreases with $l \to \infty$. We blow up the zippered rectangle by $\operatorname{diag}\big(\sqrt{\langle w(l), h\rangle^{-1}}, \sqrt{\langle w(l), h\rangle^{-1}}\big)$ to get back to unit area. Figure 3 shows a polygonal picture of the quadratic differential defined by the zippered rectangle $(\pi, w(l), h)$. The rescaled quadratic differential $q(l)$ belongs to the stratum $\mathcal{Q}^1 = \mathcal{Q}^1(4, +1)$ with one singularity of cone angle 6π and orientable foliations, and, moreover, it belongs to the thick part $\mathcal{Q}^1_{\varepsilon(l)}$,

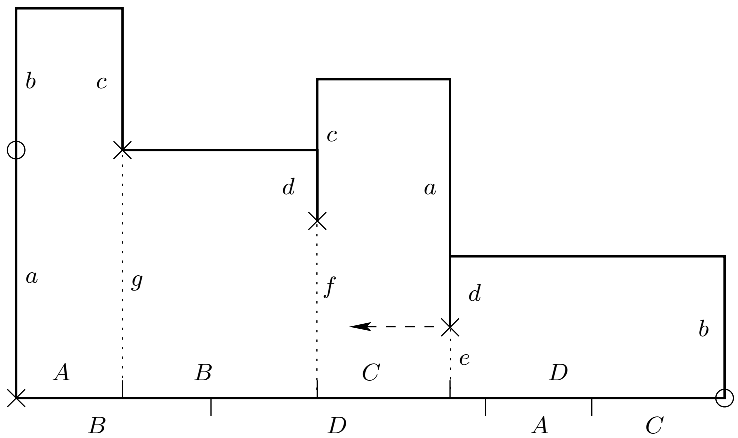

FIGURE 2. A zippered rectangle whose third subinterval degenerates. The degeneration is indicated by an arrow.

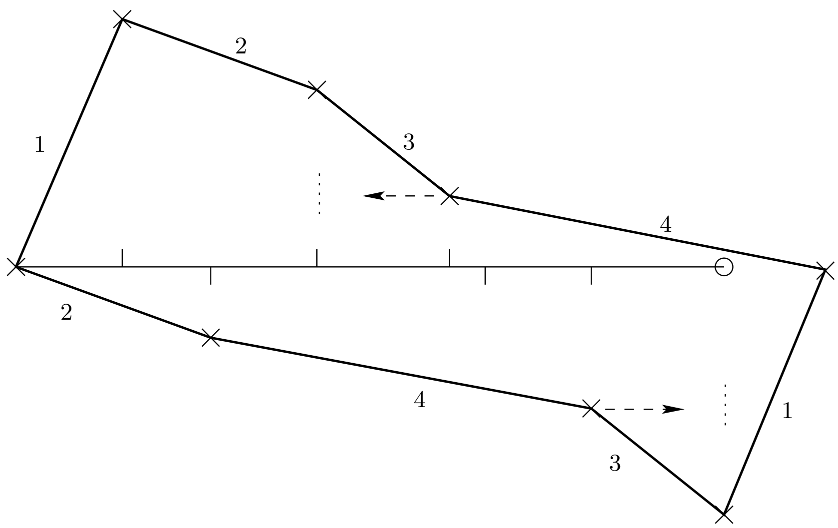

FIGURE 3. The flat metric defined by the zippered rectangle from Figure 2. Arrows indicate what happens when the zippered rectangle degenerates.

where $\varepsilon(l)$ is one tenth of the euclidean length of the side labeled 3 (Figure 3). This length decreases with $l \to \infty$ and is bounded from below by $\varepsilon(\infty) > 0$. Thus for all l we have $q(l) \in \mathcal{Q}^1_{\varepsilon(\infty)}$. In the limit we get a quadratic differential $q(\infty) \in \mathcal{Q}^1_{\varepsilon(\infty)}$ with the same polygonal pattern as in Figure 3, just the side labeled 3 will be a vertical saddle connection. The zippered rectangle in the flat metric of $q(\infty)$ is as in Figure 4. The effect of the degeneration of the zippered rectangles is this vertical saddle connection on the side labeled 3. This can not be seen in terms of the stratum.

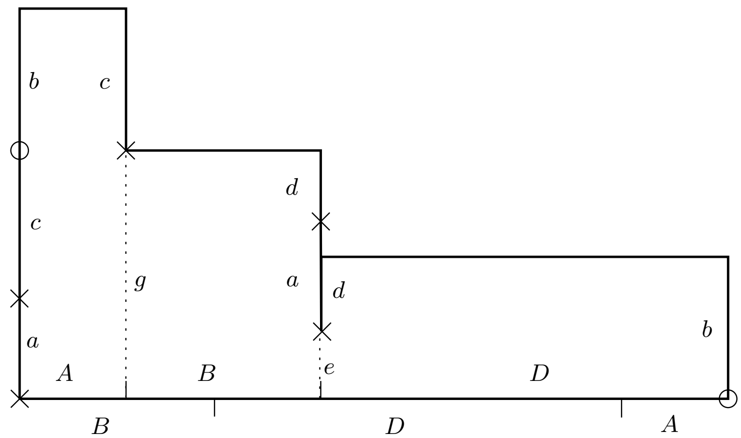

FIGURE 4. The zippered rectangle in the flat metric of the limit quadratic differential

By choosing different sequences $w(l)$ of widths we can control the vertical foliation of $q(\infty)$. Let $w_j(\infty)$, $j \in \{A, B, D\}$, denote the widths of the zippered rectangle on $q(\infty)$, c.f. Figure 4. If the widths are pairwise rationally dependent, the vertical foliation of $q(\infty)$ decomposes into a union of periodic cylinders. On the other hand, if the widths are linearly independent over $\mathbb{Q}$, the vertical foliation is minimal. Moreover, the vertical foliation will be uniquely ergodic since interval exchange transformations on three subintervals are rotations of the circle. Thus a degeneration of the zippered rectangles which define a knorke pair can lead to an asymptotic pair of Teichmüller geodesics.

The example does not prove that it is impossible to find a zippered rectangle with data inside a fixed compact set for every point in the thick part of the stratum. What the example shows is that, if this is possible, one has to choose the base interval carefully: A not-so-long saddle connection causes trouble if it does not cross the base interval and its horizontal length is small compared to the length of the base interval. Example 2 shows that saddle connections of this kind can be problematic.

We have a closer look at the Teichmüller rays $\rho_{q(l)}$. The side labeled 3 is a saddle connection which gets more and more vertical. Let's call this saddle connection $\gamma(l)$. The vertical $\ell_{q(l)}$-length is bounded from above and below independent of l, thus if we apply the Teichmüller flow $\operatorname{diag}(e^{t/2}, e^{-t/2})$, the minimal length of $\gamma(l)$ along $\rho_{q(l)}$ tends to zero. For l large enough, the Teichmüller ray leaves $\mathcal{Q}^1_{\varepsilon(\infty)}$ eventually. We will prove that Teichmüller rays always leave the thick part after a certain amount of time when the zippered rectangles have data outside a compact set. The amount of time needed turns out to depend purely on $\varepsilon(\infty)$, the length of the base interval and the number of subintervals, and it is independent of the vertical foliation. Phrased in a positive way, we prove that a zippered rectangle has uniformly bounded data if the corresponding Teichmüller ray does not leave the thick part for a certain amount of time. This threshold does not corrupt the main result since we get in any case a threshold in time (see Corolary 2.6).

Example 2: Too many not-so-long nearly vertical saddle connections.
Fix a stratum $\mathcal{Q}^1$ of orientable quadratic differentials and a small $\varepsilon > 0$. We give a sequence of quadratic differentials $q(l) \in \mathcal{Q}^1_\varepsilon$ with uniquely ergodic vertical foliation, such that for every choice of $0 < \delta < \varepsilon/2$ the widths of zippered rectangles on $q(l)$ leave every compact set $K \subset (0,\delta)^n$, independent of the horizontal intervals of admissible length we choose as base intervals.

Let $-q_0 \in \mathcal{Q}^1_\varepsilon$ be a *Strebel differential*, i.e. the vertical foliation is just one periodic cylinder. Multiplication of a quadratic differential by -1 is a rotation of the flat metric by 90 degree, therefore the horizontal foliation of $q_0 \in \mathcal{Q}^1_\varepsilon$ is just one periodic cylinder. The core curve of which shall be of length at most $1/\varepsilon$. (Choose $\varepsilon > 0$ small enough.) Thus every arc which crosses the cylinder transversally has length at least ε. Since for each quadratic differential, directions with uniquely ergodic foliations are dense on the circle, there is a sequence of angles $\theta_l \searrow 0$ such that θ_l is a uniquely ergodic direction on q_0. Thus the vertical foliation of $q(l) = e^{i(\pi - 2\theta_l)}q_0$ is uniquely ergodic and the horizontal lengths of saddle connections in direction $\pi/2 - \theta_l$ (which is the direction of the image of the Strebel cylinder) are at most $\cos(\pi/2 - \theta_l)/\varepsilon = \sin(\theta_l)/\varepsilon$. Horizontal intervals of admissible length can not cross the cylinder, hence can not cross these saddle connections. Therefore, independent of the choice of the base interval, the zippered rectangle on $q(l)$ has at least one rectangle of width at most $\sin(\theta_l)/\varepsilon$, where $\sin(\theta_l)/\varepsilon \to 0$ for $l \to \infty$.

We remark that this sequence of quadratic differentials has $-q_0$ as limit, a unit area quadratic differential whose vertical foliation has a one-cylinder-decomposition. But any zippered rectangle based at a horizontal interval of admissible length does not cover all of $-q_0$, and hence all such zippered rectangles have area strictly less than 1 – a situation as sketched right before Example 1.

Motivated by this observations our strategy is to show that zippered rectangles have data in a compact set if the initial segment of uniform length on the Teichmüller ray does not leave the thick part.

2.6. **Zippered rectangles and the thick part of the stratum.** This section is devoted to find conditions for unit area quadratic differentials to have representations by zippered rectangles with bounded widths and heights. Motivated by the above example we look for saddle connections in the zippered rectangle causing the corresponding Teichmüller geodesic to leave the thick part of the stratum if the zippered rectangle data do not belong to a certain compact set. The upshot of this section is Corollary 2.20, which states that a zippered rectangle has bounded data if the corresponding Teichmüller geodesic remains in the thick part of the stratum for at least a uniform amount of time. The main steps to prove this are as follows:

Step 1: We find a uniform lower bound on the heights of the rectangles for zippered rectangles on quadratic differentials in the thick part $\mathcal{Q}^1_\varepsilon = \mathcal{Q}^1_\varepsilon(\kappa_1, \cdots, \kappa_k, +1)$.

Step 2: We show that rectangles with large heights are glued to rectangles with large heights.

Step 3: Within rectangles with large heights we find saddle connections which get very short along the Teichmüller geodesic after a uniform amount of time. This leads to a uniform upper bound on the heights of zippered rectangles on Teichmüller geodesics which remain in the thick part of the stratum for a long time.

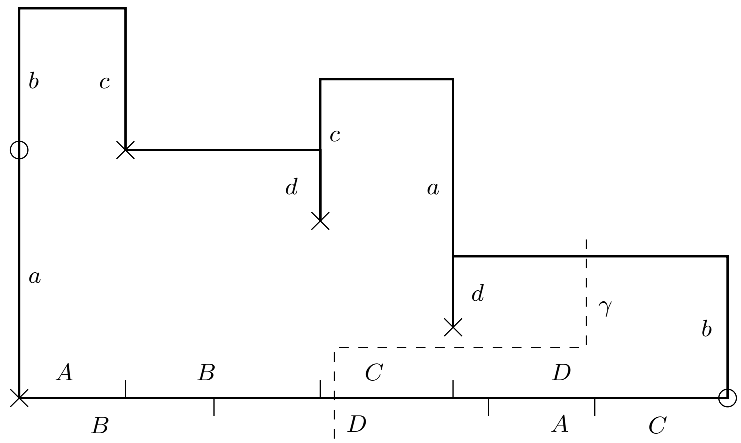

FIGURE 5. The curve γ crosses exactly one rectangle vertically.

Step 4: Using the upper bound on the heights from step 3 and the same argument again (applied to another class of saddle connections) we get uniform lower and upper bounds on the widths.

Remark. In general, step 4 implies step 3. Our argument for step 4 uses step 3.

Throughout this section we assume the vertical measured foliations to be without saddle connections and all zippered rectangles to have the maximal possible number of rectangles. This number only depends on the stratum:

$$n = |\mathcal{A}| = 2g - 1 + (\text{number of singularities}) \geq 4.$$

Recall that the vertical sides of the zippered rectangles are unions of linearly glued subintervals which we call *glueing parts* (Section 2.1). One can tell how many glueing parts are contained in each vertical rectangle side: Exactly one right vertical side consists of one glueing part, all other right vertical sides consist of two glueing parts. For the left vertical sides, either the same is true or there is one left vertical side with three glueing parts, two vertical sides with one glueing part, and all other consist of two glueing parts. In particular there are in total $2n - 1$ glueing parts, each occurring once on a left side of a rectangle and once on a right side.

2.6.1. *Step 1: A lower bound on the heights.* Our first step is to find a lower bound on the heights of zippered rectangles in the thick part of the stratum. Recall that, for given $\delta > 0$, an interval is of admissible lengths if the length of its horizontal realization is between δ and $\delta/2$.

Lemma 2.14. *Let $\varepsilon > 0$ be given and let $0 < \delta = \delta(\varepsilon) \leq \varepsilon/5$ be small compared to ε. There exists an uniform lower bound for the heights of the zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$, where $(X, \mathcal{F}) \in \mathcal{Q}^1_\varepsilon$ and I is a transversal interval of admissible length.*

Proof. Let R be the rectangle with the smallest height, denoted by h_R. Let γ be the simple closed curve crossing R vertically and closing up along I, c.f. Figure 5 (In this figure, the rectangle R is the rectangle labeled D). The length of I is at most δ, hence the ℓ_q-length of γ is bounded from above by $h_R + \delta$. Thus there is a saddle connection of ℓ_q-length at most $h_R + \delta$. Recall that ℓ_q^*-length and ℓ_q-length are the same up to a multiplicative error of at most 2. Therefore $h_R + \delta \geq \varepsilon/2$ which gives a lower bound $\varepsilon/2 - \delta \geq 3\varepsilon/10$ on the heights of rectangles. $\square$

We just found a uniform lower bound on the heights. The upper bound needs more work, and we have to require that the Teichmüller ray spends a certain amount of time in the thick part. Suppose that the zippered rectangle has some rectangles which are significantly taller than the other rectangles. These tall rectangles are glued together along their vertical sides. Note that by the area-one condition tall rectangles have to be very narrow. If there are tall rectangles, we find saddle connections with small horizontal length and bounded but large vertical length. These saddle connections shrink for a certain amount of time along the Teichmüller ray before they reach their very small length minimum. Estimating this time and the length minimum and assuming that the Teichmüller ray stays in the thick part of the stratum for a long time, we obtain a contradiction, which in turn guarantees the existence of upper bounds for the heights of the rectangles.

2.6.2. *Step 2: Rectangles with large heights are glued to rectangles with large heights.* The second step is to give a precise definition of "tall" and to show that "tall" rectangles are glued to "tall" rectangles. To this end we first fix some notation: For positive numbers m, M and δ we define a smooth increasing function

$$\tilde{H}_{m,M,\delta} : \mathbb{R} \to \mathbb{R}, \quad c \mapsto 2m\, c^{M+1}/\delta.$$

The following proposition tells us that, for c large, zippered rectangles with a rectangle of height at least $\tilde{H}_{m,M,\delta}(c)$ have the above mentioned property that "tall" rectangles are glued together.

Proposition 2.15. *Let $\varepsilon > 0$ and $\delta > 0$ be as above and let $\mathcal{Q}^1$ be a stratum of orientable unit area quadratic differentials. Let $(X, \mathcal{F}) \in \mathcal{Q}^1$ be without vertical saddle connections and let I be a transversal interval of admissible length. Thus there are in total $M = 2n - 1$ glueing parts on $\mathfrak{zr}(X, \mathcal{F}, I)$ and there are at most $m = 3$ glueing parts on each vertical rectangle side. Let $c > m + 1$.*
If $\mathfrak{zr}(X, \mathcal{F}, I)$ contains a rectangle of height at least $\tilde{H}_{m,M,\delta}(c)$, we can find an $H > 0$ with $2c/\delta \leq H \leq \tilde{H}_{m,M,\delta}(c)$ such that the length of every glueing part is either at most H/c or at least H. In other words, there isn't any glueing part α on $\mathfrak{zr}(X, \mathcal{F}, I)$ with $H/c < \ell_{(X,\mathcal{F})}(\alpha) < H$.

Proof. Let $d = 2/\delta$. As the total area of the zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$ is one, there is at least one rectangle of height at most d and hence at least one glueing part of length at most d. The total number of glueing parts is M and there are at most m glueing parts on each vertical side. Hence, for any $c > m + 1$, if there is a rectangle of height at least dmc^{M+1}, the longest glueing part on any side of this rectangle has length at least dc^{M+1}. As there are at most M glueing parts and as the ratio of the lengths of the longest and the shortest glueing part is at least c^{M+1}, by the pigeon hole principle there is a number $2c/\delta \leq H \leq dmc^{M+1}$ such that every glueing part is either longer than H or shorter than H/c. $\square$

If there were vertical saddle connections, the proposition would still be true. The proof remains literally the same, one just has to replace M and m by appropriate constants with the same properties.

Remark. Even for fixed m, δ and c, the threshold $\tilde{H}_{m,M,\delta}(c)$ heavily depends on the stratum: It depends exponentially on the number M of glueing parts. This number is larger than the number of critical rays of the vertical foliation starting at a zero of the quadratic differential. By the Gauss-Bonnet-formula, this number is bounded from below by $2g-2$, where g is the genus of S.

The proposition tells us what the right definition of "tall" should be:

Definition. Choose any $c > 4$ and let $\mathfrak{zr}(X, \mathcal{F}, I)$ be a zippered rectangle with a rectangle of height at least $\tilde{H}(c) = \tilde{H}_{3,2n-1,\delta}(c)$ and without vertical saddle connections. Let H be as in Proposition 2.15. Glueing parts of length at least H are *c-long*, whereas glueing parts of length at most H/c are *c-short*. Rectangles of height at least H are *c-tall*, whereas rectangles of height at most $3H/c$ are *c-small*.

Remark. We strongly emphasize that H depends on $\mathfrak{zr}(X, \mathcal{F}, I)$, but it is uniformly bounded by $2c/\delta$ and $\tilde{H}(c)$. Thus Proposition 2.15 just guarantees the existence of a distinction into c-tall and c-small rectangles (Without rectangles of intermediate height!) if $\mathfrak{zr}(X, \mathcal{F}, I)$ has at least one rectangle of very large height. The proposition does not give the critical height explicitly.

By Proposition 2.15 and the choice of $c > 4$, every c-tall rectangle has at least one c-long glueing part on either side, and c-small rectangles don't have c-long glueing parts. Hence, c-tall rectangles are glued to c-tall rectangles. Moreover every glueing part is either c-long or c-short and thus every rectangle is either c-tall or c-small if the zippered rectangle has at least one rectangle of height at least $\tilde{H}(c)$.

Definition. A maximal subset (maximal with respect to inclusion) of the set of c-tall rectangles, subject to the condition that they are transitively glued together along c-long glueing parts that are the lowest glueing parts of the respective sides – i.e. these c-long glueing parts are adjacent to the top side of I – will be called a *block*. A block is allowed to consist of just one c-tall rectangle.

Remark. The condition for a set of c-tall rectangles to form a block can be rephrased to be a maximal set with the property that there exists an euclidean embedded rectangle Q in the union of the c-tall rectangles such that the lower side of Q is contained in the base interval I, the left side of Q is contained in the left side of the leftmost rectangle in the block and the right side of Q is contained in the right side of the rightmost rectangle in the block.

Note that every block is a strict subset of the set of all rectangles as there have to be c-small rectangles if there is a distinction into c-tall and c-small rectangles.

2.6.3. *Step 3: An upper bound on the heights.* We proceed with Step 3: Supposed the zippered rectangle has a c-tall rectangle, find a nearly vertical saddle connection with uniformly bounded length. This saddle connection shrinks along the corresponding Teichmüller ray and causes the ray to leave the thick part of the stratum.

Proposition 2.16. *Let $\varepsilon > 0$, $\delta > 0$ be as above and let $c > 4$. Let $\mathfrak{zr}(X, \mathcal{F}, I)$ be a zippered rectangle as in Proposition 2.15 without vertical saddle connections and with a rectangle of height at least $\tilde{H}(c)$, thus a zippered rectangle whose rectangles are either c-tall or c-small and whose glueing parts are either c-long or c-short. Then there is at least one saddle connection γ with* $\mathrm{hori}(\gamma) \leq n/H$ *and* $\mathrm{vert}(\gamma) \leq 2nH/c$, *where $n = |\mathcal{A}| \geq 4$.*

This proposition is based on the following observation: Suppose there is just one c-tall rectangle, which therefore has to have a self-glueing along the vertical sides. By the area condition it has to have small width. Thus, there is a non-vertical cylinder in the c-tall rectangle such that the core curve has small horizontal length, and the vertical length is bounded from above by the lengths of the c-short glueing parts on the vertical sides. If there is a lower bound on the length of saddle connections, we get a lower bound on the vertical length as well. Thus the cylinder has small width and is nearly vertical, where the angle to the vertical is bounded from above in terms of c. The core curve gets short along the Teichmüller geodesic and the Teichmüller geodesic leaves the thick part of the stratum.

When there is more than one c-tall rectangle the situation becomes more difficult, but still there are saddle connections with similar properties. Before we prove the proposition, we give a definition to simplify notations.

Definition. Let α be a glueing part on a vertical side of a zippered rectangle. Considered as an interval the glueing part α has two endpoints. We call the upper endpoint (in the orientation of the vertical foliation) the *head* of α. We call the lower endpoint the *foot* of α.

Now we are ready to prove Proposition 2.16.

Proof of Proposition 2.16. The area of each rectangle is at most one, hence c-tall rectangles have width at most $1/H$. For the proof of the proposition we extract different types of blocks and look carefully at the glueings of the corresponding rectangles. For either type of block we find a saddle connection with the desired properties.

First type of blocks: There is a block with the property that the lowest c-long glueing part on its right side is a glueing part whose foot is a zero of the underlying orientable quadratic differential $(X, \mathcal{F})$. We define a simple arc connecting two zeros of the quadratic differential by drawing it on the zippered rectangle. We start at that foot moving horizontally to the left. On every vertical side, there are at most three glueing parts, hence below the foot there are at most two glueing parts. These glueing parts are c-short glueing parts and their lengths are at most H/c each. Thus the vertical distance between the base interval I and the horizontal line which we draw is at most $2H/c$. We cross all c-tall rectangles of the block and eventually we will hit the left end of the block. Two cases may arise:
Either the lowest c-long glueing part on the left side of the block has a foot which is a zero. (Note that the lowest glueing part of the leftmost rectangle of $\mathfrak{zr}(X, \mathcal{F}, I)$ has a zero as its foot: the left endpoint of I.) Again, below the lowest c-long glueing part there are at most two c-short glueing parts on the vertical side and each c-short glueing part is of length at most H/c. Thus we can complete a simple arc connecting two zeros by moving vertically up or down for at most $2H/c$, see Figure 6.
Otherwise at the left end of that block the lowest c-long glueing part α has a foot which is a regular point. The only regular point which can be the foot of a glueing part which is not the lowest glueing part of a side is the copy of the right endpoint of I. (In Figure 5 this point is indicated by the circle on the leftmost vertical side.) Hence the c-long glueing part α is glued to the lowest glueing part (which has to be c-long) on the right side of a block which ends at the right endpoint of I. Move up or down for at most $2H/c$ at the left end of the initial block and cross the c-long

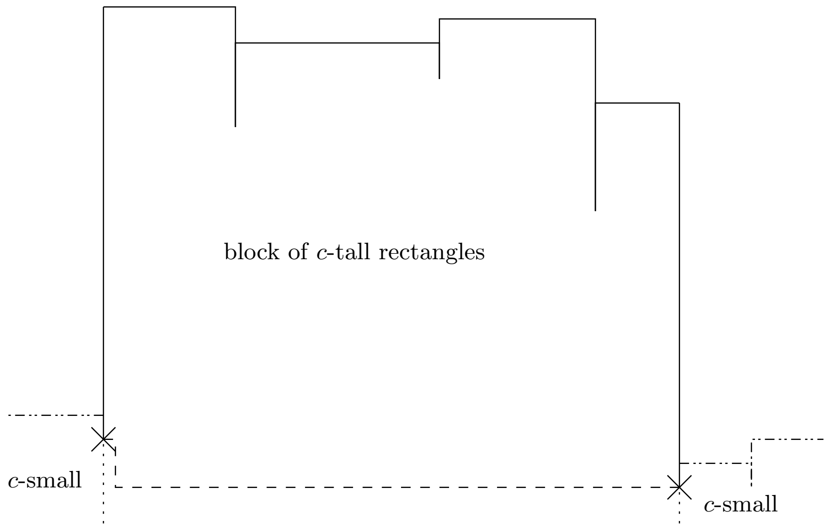

FIGURE 6. A simple arc connecting two zeros and staying close to the base interval.

glueing part with a regular point at the bottom to enter that new block at the far right of I. From there on we continue moving horizontally to the left as above. The lowest c-long glueing part on the left side of this block is a c-long glueing part which foot is a zero. Connect this zero as above.
Either way, we get a simple arc which connects two zeros, horizontally crosses every rectangle at most once and has a vertical length which is bounded from above. Pulling tight in the flat metric[13] we obtain a saddle connection γ. This saddle connection has bounded horizontal and vertical length:

$$\text{hori}(\gamma) \leq n/H \text{ and } \text{vert}(\gamma) \leq 4H/c.$$

Second type of blocks: The remaining case, i.e. the foot of every lowest rightmost c-long part of every block is a regular point for the quadratic differential. This occurs precisely when there aren't any rectangles of $\mathfrak{zr}(X, \mathcal{F}, I)$ at the right of any block and the lowest glueing part on the right side is a c-long glueing part. Thus there is just one block, and this block is as far right on I as possible. By definition there aren't any c-tall rectangles outside that block. At first we show by contradiction that there has to be a c-tall rectangle which upper right glueing part is c-short and has a zero as foot.
Label the rectangles by $\{1, \ldots, n\}$ according to their position from left to right, i.e. $\pi^0 = \text{id}$. Let $\pi^1 : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation on the subintervals, let $\{l, \ldots, n\}$ be the indices of the rectangles in the block and let $r = (\pi^1)^{-1}(n)$. Suppose that the upper glueing part on the right side of every rectangle in the block is c-long. As the upper c-long right glueing parts are glued to c-long left glueing parts, which only occur on the sides of c-tall rectangles, and as there is only one block, every rectangle R_j, $j \in \{l, \ldots, n\}$, $j \neq r$, is glued along the upper glueing part on the right side to a c-long glueing part on the left side of a rectangle of that

[13] This is done with respect to the euclidean length function $\ell^*_{(X,\mathcal{F})}$.

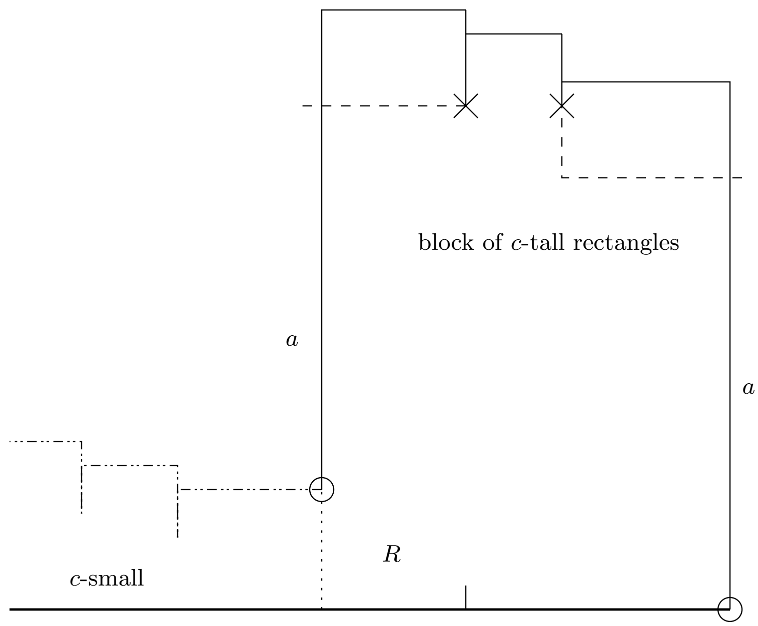

Figure 7. A simple arc connecting two zeros, following the upper part of the c-tall rectangles. The horizontal glueing is a to a.

block. Hence the rectangle R_r glued to I from below in the rightmost position has to be c-long: $l \leq r \leq n$. On the lower side of I stepping from R_r to the left rectangle by rectangle we meet every c-tall rectangle from the block before any c-small rectangle shows up. Thus π^1 has to fix the two subsets $\{1, \dots, l-1\} \sqcup \{l, \dots, n\}$, contradicting the assumption that there aren't any vertical saddle connections. Thus, in that block there is a c-tall rectangle R such that the upper right glueing part is c-short.

The foot of this glueing part has to be a zero, as the lowest right glueing part of the zippered rectangle is c-long in the case which we consider now. Starting from that zero we move horizontally to the left (c.f. Figure 7). Every time we hit the left side of a c-tall rectangle we check whether or not the upper left c-long glueing part has a zero as head. If it is a zero, connect it by moving up or down vertically at most $2H/c$ – the above argument on the number of the c-short glueing parts applies here, too. If the head isn't a zero, move up or down for at most $2H/c$ and cross the upper c-long glueing part on the left side to enter another c-tall rectangle $\tilde{R}$ via the upper right c-long glueing part of $\tilde{R}$ and continue moving horizontally to the left as above. As there are only finitely many c-tall rectangles, at some point we have to hit a c-long glueing part whose head is a zero. This happens when we get a combinatorial cycle on the set of the c-long glueing parts, for latest. Ending in that zero we draw a simple arc connecting two singularities. We pull it tight and get a saddle connection γ. This saddle connection crosses every c-tall rectangle at most once from the right to the left and gains vertical length of at most $2H/c$ every time a rectangle is crossed:

$$\mathrm{hori}(\gamma) \leq n/H \text{ and } \mathrm{vert}(\gamma) \leq 2nH/c.$$

This finishes the proof. □

The saddle connections we just found will shrink along the initial Teichmüller segment. We compute a number $T_h > 0$ such that the minimal lengths of the saddle connections are less than ε along the initial Teichmüller segment of length T_h.

Lemma 2.17. *Let $\varepsilon > 0$ and let $\delta > 0$ be as in Lemma 2.14. For every stratum $\mathcal{Q}^1$ of orientable unit area quadratic differentials there exist $c > 4$ and $T_h > 0$ with the following property:*
Let $\mathfrak{zr}(X, \mathcal{F}, I)$ be a zippered rectangle such that $(X, \mathcal{F}) \in \mathcal{Q}^1_\varepsilon$ is without vertical saddle connections and such that at least one rectangle is of height at least $\tilde{H}(c)$. Then the initial segment of length T_h on the corresponding Teichmüller geodesic $\rho_{X,\mathcal{F}}$ leaves the thick part $\mathcal{Q}^1_\varepsilon$.

Proof. We will make use of the saddle connections mentioned in Proposition 2.16. These saddle connections turn out to be mostly vertical, hence they will shrink along $\rho_{X,\mathcal{F}}$ for a long time before reaching their length minimum. Our goal is to show that there is a constant $T_h > 0$ such that up to time T_h the length of these saddle connections will be nonincreasing and, moreover, the minimal length along the initial segment of length T_h on the Teichmüller geodesic will be less than ε.

The prerequisites of Proposition 2.16 are fulfilled. Let γ be one of the saddle connections given by Proposition 2.16. The $\ell_{(X_t, e^{t/2}\mathcal{F})}$-length of γ at time t along the Teichmüller ray determined by $\mathfrak{zr}(X, \mathcal{F}, I)$ is

$$\ell_t(\gamma) = \ell_{(X_t, e^{t/2}\mathcal{F})}(\gamma) = e^{t/2}\operatorname{hori}(\gamma) + e^{-t/2}\operatorname{vert}(\gamma),$$

where $\operatorname{hori}(\gamma)$ and $\operatorname{vert}(\gamma)$ are the horizontal and vertical lengths of γ in the flat metric defined by $(X, \mathcal{F})$. On $(X, \mathcal{F})$ the saddle connections given in the proposition above have bounded horizontal length $\operatorname{hori}(\gamma) \le n\kappa$ for $\kappa = 1/H \le \delta/(2c)$. This bound tends to zero as $c \to \infty$. Recall from Section 1.1.2 the inequality $\ell^*_{q'} \dot{\prec} \ell_{q'}$, independent of the quadratic differential q'. If $(X, \mathcal{F})$ is in the thick part $\mathcal{Q}^1_\varepsilon$ of the stratum, a lower bound on the vertical length of the saddle connection follows immediately: $\operatorname{vert}(\gamma) \ge \varepsilon - n\kappa$. The quotient of vertical and horizontal length measures the angle a saddle connection makes with the horizontal direction. This quotient is bounded from below: $\operatorname{vert}(\gamma)/\operatorname{hori}(\gamma) \ge (\varepsilon - n\kappa)/(n\kappa) \to \infty$ for $\kappa \to 0$, hence for $c \to \infty$.

Differentiation with respect to time shows that the minimal ℓ_t-length is achieved at time $\tau(\gamma) = \log(\operatorname{vert}(\gamma)/\operatorname{hori}(\gamma))$ and that $\ell_{\tau(\gamma)}(\gamma) = 2\sqrt{\operatorname{vert}(\gamma)\operatorname{hori}(\gamma)}$. The minimum is achieved exactly when horizontal and vertical length coincide. This motivates the name *balanced time* for the time $\tau(\gamma)$. We see that the balanced time is independent of the length of γ and only depends on the angle between the horizontal foliation and γ. The more vertical γ is, the larger $\tau(\gamma)$ will be. Hence we can bound $\tau(\gamma)$ from below by $\log((\varepsilon - n\kappa)/(n\kappa))$. Note that for two saddle connections γ and $\tilde{\gamma}$, where γ is supposed to be more vertical than $\tilde{\gamma}$ and with $\ell_0(\gamma) = \ell_0(\tilde{\gamma})$, it holds $\ell_{\tau(\tilde{\gamma})}(\gamma) \le \ell_{\tau(\tilde{\gamma})}(\tilde{\gamma})$, but the balanced times fulfill $\tau(\gamma) > \tau(\tilde{\gamma})$. Clearly the minimal length depends linearly on the initial length if we fix the direction of the saddle connection. Therefore, in order to find suitable bounds for minimal length and balanced time, it suffices to estimate the largest possible minimal length L_h and smallest possible balanced time T_h for saddle connections given by Proposition 2.16. At time T_h every saddle connection given by the proposition has length at most L_h. We calculate an upper bound for the length

$$L_h = 2\sqrt{\mathrm{vert}(\gamma)\cdot \mathrm{hori}(\gamma)} \le 2\sqrt{2nH/c \cdot n\kappa} \le 2n\sqrt{\frac{2H}{cH}} = 2n\sqrt{2/c}$$

and an lower bound for the time

$$T_h = \log\frac{\mathrm{vert}(\gamma)}{\mathrm{hori}(\gamma)} \ge \log\frac{\varepsilon - n\kappa}{n\kappa} \ge \log\frac{\varepsilon - n\delta/(2c)}{n\delta/(2c)} \ge \log\frac{2c\varepsilon - n\delta}{n\delta} > 0.$$

Hence choosing $c = c(\varepsilon, n) > 4$ large enough, there is a $T_h = T_h(\varepsilon, \delta, n, c)$ such that each Teichmüller ray starting in the thick part of the stratum leaves the thick part of the stratum not later than at time T_h if the corresponding zippered rectangle has a rectangle of height at least $\tilde{H}(c)$. □

Remark. The threshold $T_h > 0$ can be chosen to be a uniform value for a given thick part $\mathcal{Q}^1_\varepsilon$, independent of the vertical foliation of a knorke partner inside that stratum, if we fix $\delta(\varepsilon)$ small and $c = c(\varepsilon, n)$ large. In fact, T_h only depends on ε, δ and the number of subintervals. The same is true for the bound $\tilde{H}_{3,2n-1,\delta}(c(\varepsilon, n))$ on the heights.

We restate this result as a condition for zippered rectangles to have bounded heights:

Corollary 2.18. *Fix a stratum $\mathcal{Q}^1$ of orientable unit area quadratic differentials. Let $\varepsilon > 0$, $\delta > 0$, $c = c(\varepsilon, n) > 4$ and $T_h > 0$ be as above. Let $\mathfrak{zr}(X, \mathcal{F}, I)$ be a zippered rectangle such that $(X, \mathcal{F}) \in \mathcal{Q}^1_\varepsilon$ is without vertical saddle connections. If the initial segment of length T_h on the corresponding Teichmüller geodesic $\rho_{X,\mathcal{F}}$ does not leave the thick part $\mathcal{Q}^1_\varepsilon$, the heights of the rectangles of $\mathfrak{zr}(X, \mathcal{F}, I)$ are at most $\tilde{H}(c)$.*

2.6.4. *Step 4: Bounds on the widths.* Up to now we established conditions for quadratic differentials to have zippered rectangles with bounded heights. The next step is to find a lower bound on their widths. A lower bound ω on the widths immediately gives an upper bound $\delta - (n-1)\omega$ on the widths. Thus, the zippered rectangle data are restricted to a compact set given that we have established a lower bound on the widths.

Lemma 2.19. *Let $\varepsilon > 0$ and let $\delta > 0$ be as in Proposition 2.14. For every stratum $\mathcal{Q}^1$ of orientable unit area quadratic differentials there exist $\omega > 0$ and $T_w > 0$ such that all widths of a zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$ are at least ω if $(X, \mathcal{F})$ is without vertical saddle connections and if the corresponding Teichmüller geodesic $\rho_{X,\mathcal{F}}$ stays in $\mathcal{Q}^1_\varepsilon$ uninterrupted up to time T_w.*

Proof. The argument we use is the same one we used to prove Corollary 2.18: If one rectangle is very narrow, we can find a saddle connection which is nearly vertical and which has bounded length. This saddle connection causes the Teichmüller ray to leave the thick part.

Let any $\omega > 0$ be given and let $c = c(\varepsilon, n)$ be as above. Assume $w_j < \omega$ for some $j \in \mathcal{A}$. We know by Corollary 2.18 that for $T_w \ge T_h$ large enough the height of R_j is at most $\tilde{H}(c)$, if $\mathfrak{zr}(X, \mathcal{F}, I)$ fulfills the above conditions. We find an arc connecting two zeros of the quadratic differential $q = (X, \mathcal{F})$ and with bounded horizontal and vertical lengths. This arc is the concatenation of the subinterval I_j and of two segments of the leafs that contain the vertical sides of R_j and have lengths bounded from above. We only have to show that the minimal vertical distance from an endpoint of I_j to a zero of q is bounded from above. There are three cases.

First case: The vertical side of R_j contains a zero of the quadratic differential $q = (X, \mathcal{F})$. The lowest part on that side has length at most $\tilde{H}(c)$ and connects a zero of q to the base interval I.

The second case is if there isn't any zero of q on the right vertical side of R_j. This only happens for the rightmost rectangle or for the rectangle glued to the rightmost position, i.e. I_j is the winner or the loser. The other partner in the RV-step, i.e. the respective loser or winner, must have a zero on the right vertical side. The union of the right sides of the winner and the loser is an interval contained in one leave of the vertical foliation. Hence the length of a vertical segment from the right endpoint of I_j to a zero is at most $2\tilde{H}(c)$.

The third case is if there isn't any zero on the left vertical side of R_j. This never happens for the leftmost rectangle – the left endpoint of I is a zero –, hence there is a rectangle left of R_j. This rectangle has a zero on the right side or it is a rectangle fitting into case two. Either way, we obtain a vertical segment from the left endpoint of I_j to a zero of vertical length at most $2\tilde{H}(c)$.

Connect the vertical segments which start from the left respectively from the right endpoint of I_j by the subinterval I_j and pull tight. We end up with a saddle connection with horizontal length at most ω and vertical length at most $4\tilde{H}(c)$. Recall that every saddle connection has length at least ε. Hence we found a saddle connection γ with $\varepsilon - \omega \leq \mathrm{vert}(\gamma) \leq 4\tilde{H}(c)$ and $\mathrm{hori}(\gamma) \leq \omega$. Therefore we can find an upper bound for the minimal length

$$L_w = 2\sqrt{\mathrm{vert}(\gamma)\,\mathrm{hori}(\gamma)} \leq 4\sqrt{\tilde{H}(c)\,\omega} \to 0$$

as well as a lower bound on the time at which the above saddle connection is shorter then L_w:

$$T_w = \log \frac{\mathrm{vert}(\gamma)}{\mathrm{hori}(\gamma)} \geq \log \frac{\varepsilon - \omega}{\omega} \to \infty,$$

where both limits are for $\omega \to 0$. Choosing ω small enough, there is a uniform T_w such that each Teichmüller ray starting in the thick part of the stratum will leave it not later then at time T_w if the zippered rectangle at the starting point has a rectangle of width at most ω. □

Remark. The threshold $T_w > 0$ can be chosen to be a uniform value for a given thick part $\mathcal{Q}^1_\varepsilon$, independent of the vertical foliation of a knorke partner inside that stratum, if we fix $\omega = \omega\big(\tilde{H}(c(\varepsilon, n)), \varepsilon\big)$ small enough. Recall that $\tilde{H}(c(\varepsilon, n)) = \tilde{H}_{3,2n-1,\delta}(c(\varepsilon, n))$ only depends on ε, δ and the number of subintervals, hence so does the bounds on the widths. Thus T_w can be chosen to be completely determined by ε, δ and the number of subintervals.

Corollary 2.20. *Let $\varepsilon > 0$ and let $\delta > 0$ be as in Proposition 2.14. For every stratum $\mathcal{Q}^1$ of unit area quadratic differentials there exist a compact set $K \subset (0, \delta)^n \times \mathbb{R}^n_{>0}$ and $T_w = T_w(\varepsilon, \delta, n) > 0$ such that the pair (w, h) of the vectors of widths and heights of a zippered rectangle $\mathfrak{zr}(X, \mathcal{F}, I)$ is contained in K if $(X, \mathcal{F})$ is a knorke partner and if the corresponding Teichmüller geodesic $\rho_{X,\mathcal{F}}$ stays in $\mathcal{Q}^1_\varepsilon$ uninterrupted up to time T_w.*

Definition. Let $\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1)$ be the set of all quadratic differentials q in the thick part $\mathcal{Q}^1_\varepsilon$ of the stratum $\mathcal{Q}^1$ subject to the condition that the initial segment of length $T = T_w + T_c$ on the corresponding Teichmüller ray ρ_q is contained in $\mathcal{Q}^1_\varepsilon$, where $T_w = T_w(\varepsilon, \delta, n)$ is as in Corollary 2.20, and T_c as in Lemma 2.12 depends on δ, the compact set K and the number of subintervals n.

2.7. **Exponential asymptotics.** In the above sections we found an upper bound for Teichmüller distance in terms of zippered rectangles, examined its behavior along Teichmüller segments that give rise to zippered rectangles with bounded data, and established conditions on the quadratic differentials for giving rise to zippered rectangles with bounded data. We put together these results and get exponential asymptotics for knorke pairs. For a precise formulation we fix some notation first:

Definition. Let $\varepsilon > 0$ and a stratum $\mathcal{Q}^1$ be given. For every quadratic differential q in the stratum $\mathcal{Q}^1$ define

$$T_{\varepsilon,q}(t) = \lambda\{s \in [0,t] : \rho_q(s) \in \mathcal{Q}_{good}(\varepsilon, \varepsilon/10, \mathcal{Q}^1)\}.$$

Here, λ denotes Lebesgue measure on the reals.

The following lemma is the main consequence of the results we proved above. Below we will give theorems putting the lemma into context.

Lemma 2.21. *Let $\mathcal{F}$ be a minimal measured foliation that can be realized without saddle connections and let $\varepsilon > 0$. There exists a constant $\xi > 0$ depending on ε and the stratum $\mathcal{Q}^1$ of the realization of $\mathcal{F}$ such that for every knorke pair $(X, \tilde{X})$ in direction $\mathcal{F}$ there are constants $D > 0$ and $d > 0$ such that, under the additional assumption that the corresponding magnitudes (as in Section 2.4) will eventually be small enough along the Teichmüller ray $(X, \mathcal{F})$, the Teichmüller distance along the knorke pair decreases like an exponential function:*

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}}(t), \rho_{\tilde{X},\mathcal{F}}(t)\big) \leq D \exp\big(-\xi \cdot T_{\varepsilon,(X,\mathcal{F})}(t)\big)$$

for $t \geq d$.

Proof. If $\mathcal{F}$ is non-orientable, we pass to the orientation double cover and work with the knorke pair $(\mathrm{cov}_{\mathcal{F}}(X), \mathrm{cov}_{\mathcal{F}}(\tilde{X}))$ in direction $\mathrm{cov}_{\mathcal{F}}(\mathcal{F})$ (compare Theorem 1.9 and the remark after that theorem). At the end of this proof we will get information on the distance of the orientation double covers, but this information projects down to information on the Teichmüller distance along the knorke pair $(X, \tilde{X})$, c.f. Lemma 1.7, Corollary 1.8 and Proposition 1.10. Thus, without loss of generality we can assume $\mathcal{F}$ to be orientable.
If we can find long segments on the Teichmüller rays such that the widths and heights of the zippered rectangles are bounded from above and below, and that the magnitude of the height differences is small, Lemma 2.12 will give the desired decreasing behavior of Teichmüller distance. The constant ξ depends on the constants in Lemma 2.12. We assume that the magnitudes of the height differences (compare Section 2.4) will eventually start being small enough along the Teichmüller ray $\rho_{X,\mathcal{F}}$. The constant d in the theorem takes care of this threshold in time. Hence we are done if we can control the heights and widths of the zippered rectangles. Let $\delta = \varepsilon/10$. Note that δ is as in Lemma 2.14 a bound for admissible length in the ε-thick part $\mathcal{Q}^1_\varepsilon$ of the stratum. Let I be a transversal interval of admissible length on $(X, \mathcal{F})$ and $(\tilde{X}, \mathcal{F})$, and let T_c be as in Lemma 2.12. Corollary

2.20 tells us that along the Teichmüller segment $\rho_{X,\mathcal{F}}([t, t+T_c])$ the zippered rectangles based on intervals of admissible length have bounded widths and heights as long as $(X_t, \mathcal{F}_t)$ belongs to $\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1)$. But we may not just measure how long the knorke partner $(X, \mathcal{F})$ stays in $\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1)$, as Lemma 2.12 guarantees the uniform decreasing factor only along disjoint Teichmüller segments of length T_c. Hence we have to count how many disjoint segments we can find. In other words, we may count a 1 every time $(X_t, \mathcal{F}_t)$ enters $\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1)$, and we may add an additional 1 if the knorke partner spends a time span of length T_c in $\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1)$ uninterrupted. This sum is bounded from below by $T_{\varepsilon,(X,\mathcal{F})}(t)/T_c$ (in fact, if the knorke partner leaves $\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1)$ immediately every time it enters this set, that bound will be a very bad bound). Note that T_c does not depend on the knorke pair and hence can be put into the constant ξ. The constant D depends on Teichmüller distance $d_{\mathscr{T}}(\rho_{X,\mathcal{F}}(d), \rho_{\tilde{X},\mathcal{F}}(d))$ at time $t = d$. □

Masur proved that for every uniquely ergodic measured foliation $\mathcal{F}$, knorke pairs are positively asymptotic ([Mas80] and Section 1.2.3). The prove of Masur's theorem gives even more: Along the Teichmüller rays the magnitudes will converge to zero. Keeping Corollary 1.8 in mind we can reformulate the above result.

Theorem 2.22. *Let $\mathcal{F}$ be a minimal measured foliation and let $\varepsilon > 0$. There exists a constant $\xi > 0$ depending on ε and the stratum of the realization of $\mathcal{F}$ such that for every knorke pair $(X, \tilde{X})$ in direction $\mathcal{F}$ which returns to the ε-thick part of the stratum for arbitrarily large times there are constants $D > 0$ and $d > 0$ such that the Teichmüller distance along the knorke pair decreases like an exponential function:*

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}}(t), \rho_{\tilde{X},\mathcal{F}}(t)\big) \leq D \exp\big(-\xi \cdot T_{\varepsilon,(X,\mathcal{F})}(t)\big)$$

for $t \geq d$.

One might consider this theorem being sightly unaesthetic: Given a quadratic differential $(X, \mathcal{F})$ at random choice we don't have any a priori information on the amount of time the Teichmüller ray will spend in $\mathcal{Q}_{good}$. Using ergodicity of the Teichmüller flow we can prove a result that does not involve $\mathcal{Q}_{good}$ explicitly. To this end recall Birkhoff's Ergodic Theorem (see [BM00] for instance).

Theorem 2.23. *Let (Y, ν) be any probability space and let ϕ_t be a continuous flow on Y that preserves ν and is ergodic with respect to ν, i.e. any ϕ_t-invariant set $A \subset Y$ has measure $\nu(A) = 0$ or $\nu(Y \setminus A) = 0$. For any $f \in L^1(Y, \mathbb{R})$ and ν-almost every $y \in Y$ time average converge to space average:*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\phi_s y) ds = \int_Y f(\zeta) d\nu(\zeta)$$

Note that the right hand side equals $\nu(A)$ for $f = \chi_A$ the characteristic function on a Borel set $A \subset Y$.

Denote the projection from strata over Teichmüller space into strata over moduli space by $\pi : \mathcal{Q}^1 \to \mathscr{Q}^1$. Let $\mathscr{Q}_{good}(\varepsilon, \delta, \mathscr{Q}^1) = \pi(\mathcal{Q}_{good}(\varepsilon, \delta, \mathcal{Q}^1))$ be the set of all quadratic differentials over moduli space that define geodesics whose initial segment of length $T = T_w + T_c$ is contained in $\mathscr{Q}^1_\varepsilon$. For a geodesic in moduli space, defined by some quadratic differential $\pi(X, \mathcal{F})$, we set

$$T_{\varepsilon,\pi(X,\mathcal{F})}(t) = \lambda\{s \in [0,t] : \pi(\rho_{X,\mathcal{F}}(s)) \in \mathscr{Q}_{good}(\varepsilon, \varepsilon/10, \mathscr{Q}^1)\}.$$

For every $t > 0$ it holds $T_{\varepsilon,\pi(X,\mathcal{F})}(t) = T_{\varepsilon,(X,\mathcal{F})}(t)$.

Let $\mathcal{Q}^1$ be a stratum of unit area quadratic differentials over Teichmüller space. Masur [Mas82a] and Veech [Vee82] constructed measures $\tilde{\nu}$ on $\mathcal{Q}^1$ which are preserved by and ergodic with respect to Teichmüller flow. These measures give infinite volume to the stratum. To get probability measures we can pass to moduli space. The projection of the measure $\tilde{\nu}$ on $\mathcal{Q}^1$ to a measure on $\mathscr{Q}^1$ is of finite volume, thus we can rescale it to get a probability measure ν (see [Mas82a, Proposition 5.1] for instance).

Let ν be a Teichmüller flow-invariant and ergodic probability measure on the stratum $\mathscr{Q}^1$ over moduli space. For all subsets $A \subset \mathscr{Q}^1$ with measure $\nu(A) > 0$, for ν-almost all quadratic differentials $\pi(X, \mathcal{F})$ and for all small $0 < \alpha < 1$, Birkhoff's theorem implies the existence of a $T^* > 0$ such that for all $t > T^*$, we have

$$\int_0^t \chi_A(\pi(X_s, \mathcal{F}_s))ds \geq (1-\alpha)\nu(A) \cdot t. \tag{2}$$

We want to apply Birkhoff's theorem to the function $T_{\varepsilon,\pi(X,\mathcal{F})}(t)$, which is the integral of the characteristic function on $\mathscr{Q}_{good}(\varepsilon, \varepsilon/10, \mathscr{Q}^1)$ composed with Teichmüller geodesic flow. Hence we want to show that $\mathscr{Q}_{good}(\varepsilon, \varepsilon/10, \mathscr{Q}^1)$ has non-trivial measure. Let $\pi\{(X_t, \mathcal{F}_t) : t \in \mathbb{R}\}$ be any closed loop in $\mathscr{Q}^1_\varepsilon$ (thus $\{(X_t, \mathcal{F}_t) : t \in \mathbb{R}\}$ projects to a pseudo-Anosov-axis $\rho_{X,\mathcal{F}}$ in $\mathscr{T}(S)$). Obviously all tangential quadratic differentials $(X_t, \mathcal{F}_t)$ along that axis are elements of $\mathcal{Q}^1_\varepsilon$, thus $\{(X_t, \mathcal{F}_t) : t \in \mathbb{R}\} \subset \mathcal{Q}_{good}(\varepsilon, \varepsilon/10, \mathcal{Q}^1)$. As Teichmüller flow is continuous there exists a small open neighborhood $\mathcal{N}$ of the axis such that all quadratic differentials in $\mathcal{N}$ are elements of $\mathcal{Q}_{good}(\varepsilon, \varepsilon/10, \mathcal{Q}^1)$, too. The projection $\pi(\mathcal{N})$ of this set is a non-empty open subset of $\mathscr{Q}_{good}(\varepsilon, \varepsilon/10, \mathscr{Q}^1) \subset \mathscr{Q}^1$ and thus has non-trivial measure for any measure in Lebesgue measure class for instance (as the above mentioned measures due to Masur [Mas82a] or Veech [Vee82] are). Hamenstädt established the existence of an uncountable family of probability measures on the bundle of unit area quadratic differentials meeting the above properties ([Ham06]).

Theorem 2.24. *Let $\varepsilon > 0$. Let ν be any* Map*-invariant, Teichmüller flow-invariant and ergodic measure on the stratum $\mathcal{Q}^1$ that projects down to a probability measure on $\mathscr{Q}^1$. Assume that ν assigns non-zero measure to any open non-empty set. There are a constant $\xi^* > 0$ and a subset $A \subset \mathcal{Q}^1$ of co-null measure such that for every $(X, \mathcal{F}) \in A$ (where $\mathcal{F}$ will turn out to be a uniquely ergodic measured foliation without saddle connections as its realization on X) and for every $\tilde{X}$ such that $(X, \tilde{X})$ is a knorke pair in direction $\mathcal{F}$ there is a constant $D^* > 0$ such that*

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}}(t), \rho_{\tilde{X},\mathcal{F}}(t)\big) \leq D^* \exp(-\xi^* t)$$

for t larger than a certain threshold depending on X and $\tilde{X}$.

Proof. By the assumptions on ν and as $\mathscr{Q}_{good}(\varepsilon, \varepsilon/10, \mathscr{Q}^1)$ contains an open non-empty subset $\pi(\mathcal{N})$, the measure of $\mathscr{Q}_{good}(\varepsilon, \varepsilon/10, \mathscr{Q}^1)$ is not zero. Let $(X, \mathcal{F}) \in \mathcal{Q}^1$ be such that Birkhoff (in the formulation of Formula (2)) is applicable to $\pi(X, \mathcal{F}) \in \mathscr{Q}^1$ for $\alpha = 1/100$. Together with Theorem 2.22 – supposed the knorke pair $(X, \tilde{X})$ meets the conditions of this theorem – we get an upper bound

$$\begin{aligned} d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}}(t), \rho_{\tilde{X},\mathcal{F}}(t)\big) &\leq D \exp\big(-\xi \cdot T_{\varepsilon,(X,\mathcal{F})}(t)\big) = D \exp\big(-\xi \cdot T_{\varepsilon,\pi(X,\mathcal{F})}(t)\big) \\ &\leq D \exp\big(-\xi \cdot (1-\alpha)\nu(\mathscr{Q}_{good}) \cdot t\big) \\ &\leq D^* \exp\big(-\xi^* t\big) \end{aligned}$$

for $t > \max\{T^*, d\}$. Birkhoff rules out a set of measure zero in $\mathscr{Q}^1$. Since $\mathrm{Map}(S)$ is countable, the lift of this exceptional set to $\mathcal{Q}^1$ has measure zero, too. Quadratic differentials outside this exceptional set define Teichmüller rays returning to $\mathcal{Q}^1_\varepsilon$ after arbitrarily large times and thus have uniquely ergodic measured foliations as vertical foliation (Corollary 1.8). The condition on uniquely ergodic measured foliations being realized without saddle connections rules out a closed set without interior. Hence we are left with a set $A \subset \mathcal{Q}^1$ of co-null measure. □

2.8. **Example: slow asymptotics.** We give an example of a knorke pair with the property that every exponentially decreasing function asymptotically is less than the distance between the knorke partners. The knorke pair is constructed inductively. The inductive argument uses the fact that two Teichmüller rays stay bounded distance apart if the common vertical foliation decomposes into a union of periodic cylinders ([Mas75], see Section 1.2.3): If the rays are in small distance, we follow a pair of differentials with common periodic vertical foliation for a long time. The distance between the two rays then is bounded from below for a long time and eventually larger than any exponentially decreasing given function. These differentials are carefully chosen and the concatenations of the so-defined geodesic segments follow a knorke pair very closely. Hence, for a long time the knorke partners stay bounded distance apart, too.

Initial step: Let $(X, \mathcal{F}_1)$ be any Veech surface in any stratum $\mathcal{Q}^1$ of unit area quadratic differentials. Suppose that $\mathcal{F}_1$ is uniquely ergodic. Choose $(\tilde{X}, \mathcal{F}) \in \mathcal{Q}^1$ not on the Teichmüller disc defined by $(X, \mathcal{F}_1)$ such that $(X, \tilde{X})$ is a knorke pair and such that for $T_1 = 1000$ we have

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_1}(T_1), \rho_{\tilde{X},\mathcal{F}_1}(T_1)\big) > 2\exp(-T_1/1) \stackrel{\text{def}}{=} 2E_1.$$

To get the inductive step working, we define $T_0 = 0$ and $E_0 = d_{\mathscr{T}}(X, \tilde{X})$.

To simplify notation, for any $t \in \mathbb{R}$ we write

$$\mathcal{I}_t = [t - E_0, t + E_0] \subset \mathbb{R},$$

and for a point $Y \in \mathscr{T}$ and a compact set $\mathcal{C} \subset \mathscr{T}$ we write $d_{\mathscr{T}}(Y, \mathcal{C})$ to denote the minimal distance between Y and $\mathcal{C}$.

Inductive step: Let $k > 1$ be an integer. Suppose that we already constructed

(1) a sequence of uniquely ergodic directional foliations $\mathcal{F}_l$, $0 < l < k$, of the flat metric defined by $(X, \mathcal{F}_1)$, and
(2) for $0 < l < k$ an increasing sequence $T_l > T_{l-1} + 1$ of times and a decreasing sequence $0 < E_l = \exp(-T_l/l) < E_{l-1}/2$ of numbers such that the Teichmüller segments $\rho_{X,\mathcal{F}_{k-1}}([T_{l-1}, T_l])$ intersect the thick part of the stratum and

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_{k-1}}(T_l), \rho_{\tilde{X},\mathcal{F}_{k-1}}(\mathcal{I}_{T_l})\big) > \Big(2 - \sum_{j=l}^{k-1} 1/5^j\Big) E_l.$$

We construct $\mathcal{F}_k$, T_k, and E_k extending the above sequences. All directional foliations that arise are directional foliations for the flat metric defined by $(X, \mathcal{F}_1)$. Angles are measured with respect to the horizontal in this metric.

The foliation $\mathcal{F}_{k-1}$ is uniquely ergodic. Teichmüller geodesics defined by Veech surfaces with uniquely ergodic vertical foliation are recurrent. Hence there is a $\tilde{T}_k > T_{k-1}$ such that $\rho_{X,\mathcal{F}_{k-1}}(\tilde{T}_k)$ is in the thick part of the stratum.

Let $\theta_{k-1} \in \mathbb{S}^1$ be the direction of $\mathcal{F}_{k-1}$ in the flat metric defined by $(X, \mathcal{F}_1)$. The Teichmüller geodesic flow is continuous. Choose an open and connected neighborhood $\mathcal{U}_k \subset \mathbb{S}^1$ of θ_{k-1} such that for every $\theta \in \mathcal{U}_k$ with directional foliation $\mathcal{F}_\theta$ and for all $0 \le t \le \tilde{T}_{k-1} + E_0$ it holds

$$\begin{aligned} &d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_\theta}(t), \rho_{X,\mathcal{F}_{k-1}}(t)\big) < E_{k-1}/10^k \quad \text{and} \\ &d_{\mathscr{T}}\big(\rho_{\tilde{X},\mathcal{F}_\theta}(t), \rho_{\tilde{X},\mathcal{F}_{k-1}}(t)\big) < E_{k-1}/10^k. \end{aligned}$$

Periodic directions for the Veech surface $(X, \mathcal{F}_1)$ are dense in $\mathbb{S}^1$. Thus arbitrarily close to θ_{k-1} there are periodic directions for the flat metric defined by $(X, \mathcal{F}_1)$. Let the measured foliation $\mathcal{F}_k^*$ be a directional foliation given by a periodic direction in $\mathcal{U}_k$. It may happen that $(\tilde{X}, \mathcal{F}_k^*)$ is not of unit area, but it exists a point $\tilde{X}_k$ of distance at most E_0 to $\tilde{X}$ along the Teichmüller geodesic, such that $(\tilde{X}_k, \mathcal{F}_k^*)$ is of unit area. Masur's result ([Mas75] and Section 1.2.3) implies the existence of an $t_k > 1$ such that for all $t > t_k$ we have

$$\begin{aligned} d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_k^*}(\tilde{T}_{k-1}+t), \rho_{\tilde{X},\mathcal{F}_k^*}(\mathcal{I}_{\tilde{T}_{k-1}+t})\big) &> 2\exp(-T_k/k) = 2E_k \\ &> 2\exp\big(-(\tilde{T}_{k-1}+t)/k\big), \end{aligned}$$

where $T_k = \tilde{T}_{k-1} + t_k$ and $E_k = \exp(-T_k/k) < E_{k-1}/2$.

Again we make use of the continuity of the Teichmüller geodesic flow. Let $\theta_k \in \mathcal{U}_k$ be such that the foliation $\mathcal{F}_k$ in direction θ_k is uniquely ergodic and such that the Hausdorff distance between the compact Teichmüller geodesic segments $\rho_{X,\mathcal{F}_k}([0, T_k + E_0])$ and $\rho_{X,\mathcal{F}_k^*}([0, T_k + E_0])$ as well as the Hausdorff distance between $\rho_{\tilde{X},\mathcal{F}_k}([0, T_k+E_0])$ and $\rho_{\tilde{X},\mathcal{F}_k^*}([0, T_k+E_0])$ is at most $E_k/10^k$. It may happen that $(\tilde{X}, \mathcal{F}_k)$ is not of unit area. As above we move $\tilde{X}$ along the Teichmüller geodesic a distance at most E_0 to solve this issue.

For every $0 < l < k+1$ we have

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_k}(T_l), \rho_{\tilde{X},\mathcal{F}_k}(\mathcal{I}_{T_l})\big) > \Big(2 - \sum_{j=l}^{k} 1/5^j\Big) E_l.$$

Constructing the example: In the inductive step we took care that for every compact interval $I \subset \mathbb{R}_{>0}$ the Teichmüller geodesic segments $\rho_{X,\mathcal{F}_k}(I)$ converge in the Hausdorff topology. Thus the direction of the foliations converge. Let $\mathcal{F}_\infty$ be the foliation in the limiting direction. The geodesic $\rho_{X,\mathcal{F}_\infty}$ is the Hausdorff limit of the geodesics $\rho_{X,\mathcal{F}_k}$. Thus the geodesic $\rho_{X,\mathcal{F}_\infty}$ inherits recurrence from the geodesics $\rho_{X,\mathcal{F}_k}$ and, by Masur's criterion (Theorem 1.4), its vertical foliation $\mathcal{F}_\infty$ is uniquely ergodic. It may happen that $(\tilde{X}, \mathcal{F}_\infty)$ is not of unit area, but it exists a tangential unit area quadratic differential with vertical foliation $\mathcal{F}_\infty$ on the Teichmüller geodesic whose base point is of distance $|\tau| \le E_0$ to $\tilde{X}$. The pair of Teichmüller geodesics defined by $(X, \mathcal{F}_\infty)$ and $(\tilde{X}, \mathcal{F}_\infty)$ is positively asymptotic ([Mas80] and Section 1.2.3), but the distance decreases more slowly than any exponentially decreasing function:

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_\infty}(T_k), \rho_{\tilde{X},\mathcal{F}_\infty}(T_k+\tau)\big) \to 0$$

for $k \to 0$, and

$$d_{\mathscr{T}}\big(\rho_{X,\mathcal{F}_\infty}(T_k), \rho_{\tilde{X},\mathcal{F}_\infty}(T_k+\tau)\big) > 3/4\exp(-T_k/k)$$

for all $k > 0$, where $T_k > T_{k-1} + 1$.

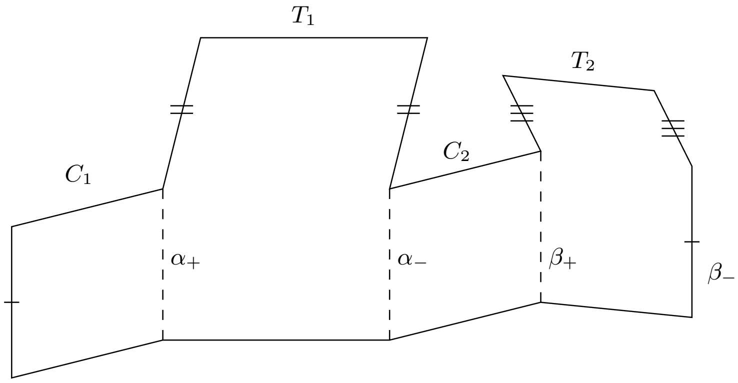

FIGURE 8. A 2T2C-splitting. Glueings are as indicated and by nearly vertical translations.

3. MINIMALITY AND NON-ERGODICITY ON A FAMILY OF FLAT METRICS

Within this chapter we have the standing assumption that all surfaces are compact without boundaries and without punctures, unless they are cylinder or slit tori. All quadratic differentials are supposed to be orientable and holomorphic. We consider strata over moduli space only.

3.1. **A family of flat metrics in genus 3.** Recall that the hyperelliptic locus $\mathscr{L} \subset \mathscr{Q}(4,4,+1)^{odd}$ and the principal stratum $\mathscr{Q}(1,1,1,1,-1)$ in genus 2 are $\mathrm{GL}(2,\mathbb{R})$-equivariantly isomorphic. The hyperelliptic locus $\mathscr{L}$ is studied by Hubert, Lanneau and Möller in [HLM07] and [HLM08]. A certain class of surfaces turned out to play a special role in this locus: so called 2T2C-surfaces. A flat metric is a *2T2C-surface* if there is a direction θ such that the foliation in direction θ decomposes into two tori and two cylinders, i.e. the complement of the saddle connections in direction θ is a union of two slit tori and two cylinders. Figure 8 shows a picture of a 2T2C-surface. Let T_1 and T_2 be the two flat tori (considered as closed surfaces without the slit) and C_1 and C_2 be the two flat cylinders in the decomposition.

Many 2T2C-surfaces do not admit a hyperelliptic involution. The condition for a 2T2C-surface to be an element of the hyperelliptic locus $\mathscr{L}$ can be phrased in the following way: If the two cylinders C_1 and C_2 represent the same class in the space of flat metrics modulo isometries preserving the horizontal direction (*marked isometries* for short), i.e. if there is an isometry between C_1 and C_2 which maps horizontal lines to horizontal lines, then the quadratic differential defining this flat metric is an element of $\mathscr{L}$ (and the hyperelliptic involution fixes the tori and exchanges the cylinders). The same statement is true if the marked isometry classes of the two tori T_1 and T_2 coincide (and the hyperelliptic involution fixes the cylinders and exchanges the tori). The following definition is borrowed from [HLM08].

Definition. A flat metric in the hyperelliptic locus $\mathscr{L}$ is a *$2T_{fix}2C$-surface* if it is a 2T2C-surface and the first condition holds, i.e. the hyperelliptic involution fixes the two tori and exchanges the two cylinders. We call the according decomposition a *$2T_{fix}2C$-splitting*.

Hubert, Lanneau and Möller call a splitting irrational if the direction of the splitting is minimal in at least one of the two tori. This is not strong enough for our needs. We make the following definition:

Definition. Consider a $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-surface. Let α_+, α_-, β_+ and β_- be the saddle connections which give the $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting into tori T_1, T_2 and cylinders C_1, C_2 as shown in Figure 8. The orientation of the saddle connections is such that the geodesics run from bottom to top in the orientation given by the vertical foliation. The *splitting vector* w is the common holonomy of these saddle connections. Let C be the common marked isometry class of C_1 and C_2. We will denote the splitting by (T_1, T_2, C, w) for short.
The splitting is called *irrational* if it has two minimal components, i.e. the direction of w (which is the direction of the splitting) is a minimal direction on both tori T_1 and T_2.

Remark. The hyperelliptic involution interchanges α_+ and α_- as well as β_+ and β_-, see [HLM07]. All four of them are simple closed curves.

Vice versa, given a tupel (T_1, T_2, C, w) such that on a cylinder in the marked isometry class C of cylinders the core curve has length and direction equal to the length and direction of the real two-dimensional vector w and such that curves in direction of w don't close up with length less or equal to $|w|$ on the tori T_1 and T_2, we can construct an oriented quadratic differential $q \in \mathcal{Q}(4, 4, +1)$ with the technique mentioned in Section 1.4.1: Slit the tori T_1 and T_2 such that the holonomy of each slit equals w, and take two copies C_1 and C_2 of a cylinder in C. Glue the cylinders and slitted tori according to the pattern shown in Figure 8. Let α be the slit on T_1 and let α^- be the left side of the slit, α^+ the right side. Denote the slit and its sides on T_2 by β, β^- and β^+.

Recall from Section 1.4.2 that the dynamics on flat metrics given by quadratic differentials in the hyperelliptic locus $\mathcal{L}$ are in correspondence with the dynamics on flat metrics given by quadratic differentials in the principal stratum $\mathcal{Q}(1, 1, 1, 1, -1)$ in genus 2. The main result of this chapter is

Theorem 3.1. *Let $q \in \mathcal{L}$ be given. If the flat metric defined by q admits an irrational $2T_{fix}2C$-splitting, then there are uncountably many minimal non-ergodic directions on q.*

The theorem provides us with an explicit criterion to check whether a quadratic differential has uncountably many minimal non-ergodic directional foliations.

Corollary 3.2. *The Arnoux-Yoccoz surface*[14] *in genus* 3 *admits uncountably many minimal non-ergodic directions.*

Proof. Let q be a quadratic differential defining the genus-3 Arnoux-Yoccoz surface. Hubert, Lanneau and Möller examined the Teichmüller disc of the Arnoux-Yoccoz surface in their paper [HLM07]. They proved that the flat metric of q admits a $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting, non-periodic in both tori (this is Lemma 5.4, especially Claim 5.5). Using the above result we conclude that there are uncountably many minimal non-ergodic directions for q. □

[14] More on the Arnoux-Yoccoz surface including a definition can be found in the paper [AY81] by Arnoux and Yoccoz.

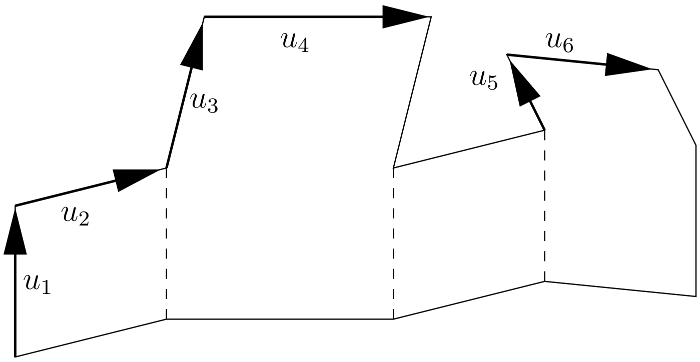

FIGURE 9. A six-tuple of saddle connections which serve as local coordinates.

A result of Masur and Smillie is that in every stratum of quadratic differentials the Hausdorff dimension of the set of nonergodic directions generically is positive ([MS91]). This doesn't give information in our setting: The hyperelliptic locus $\mathscr{L}$ is defined by equations describing the extra symmetry coming from the hyperelliptic involution. Thus it has positive codimension and measure zero.

The condition on the dynamics in direction of the splitting (i.e. non-periodicity in the tori T_1 and T_2) locally rules out a countable union of real codimension-1-submanifolds in $\mathscr{L}$: Let $u_1, \ldots, u_6$ be saddle connections as shown in Figure 9. These saddle connections serve as a set of local coordinates for a neighborhood of a $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-surface in the hyperelliptic locus $\mathscr{L}$. The direction of u_1 neither is allowed to be the direction of any vector in the lattice spanned by u_3 and u_4 nor in the lattice spanned by u_5 and u_6 (see claim in proof of Proposition 3.5). For chosen $u_2, \ldots, u_6$, both lattices exclude countably many directions for u_1.

The proof of Theorem 3.1 we are presenting is inspired by the proof Cheung and Masur gave in [CM06] for a similar result in genus 2. First we give an outline of our proof before actually proving the theorem in the following sections. The main tool is a non-ergodicity criterion by Masur and Smillie.

Theorem 3.3 ([MS91])**.** *Let (T_1^n, T_2^n, C^n, w_n) be a sequence of $2T_{\mathit{fix}}2C$-splittings of a given flat metric and assume that the directions of the vectors w_n converge to some direction θ_∞. Let $h_n > 0$ be the component of w_n perpendicular to θ_∞ and let a_n be the maximum in change of area: $a_n = \max\big(\mathrm{area}(T_1^n \Delta T_1^{n+1}), \mathrm{area}(T_2^n \Delta T_2^{n+1})\big)$. If*

- $\sum_{n=1}^{\infty} a_n < \infty$,
- *there exists $c > 0$ such that* $\mathrm{area}(T_1^n) > c$, $\mathrm{area}(T_2^n) > c$ *for all $n \in \mathbb{N}$ and*
- $\lim_{n\to\infty} h_n = 0$,

then θ_∞ is a non-ergodic direction for the flat metric.

Sketch of proof of Theorem 3.1. By an inductive process we construct uncountably many sequences of $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings meeting the condition of the criterion. Given a splitting we find a suitable cylinder such that applying powers of Dehn twists in that cylinder leads to new $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings. The change of area can be bounded from above by the sum of the areas of suitable parallelograms embedded in the splitting tori and splitting cylinders, where the cylinder of the Dehn twist is contained in the union of the parallelograms. The condition on the components of w_n perpendicular to θ_∞ can be formulated in terms of these areas, too. Using a result based on Ratner's theorem allows us to make the areas of the parallelograms as small as necessary. At this point, irrationality of the splitting is crucial. So there are two issues: First, starting with an irrational splitting, we want to end up with an irrational splitting again. This holds by a finiteness argument. Second, we have to make sure that the union of the parallelograms contains a cylinder suitable for Dehn twist of the splitting. Again, this condition can be formulated in terms of areas. These can be controlled by Ratner's theorem. $\square$

Sections 3.2 and 3.3 provide the just mentioned information we need to prove the theorem in Section 3.4.

3.2. **Splittings and twisting a splitting.** We collect some pieces of information on splittings. As mentioned above we have to find cylinders that allow to Dehn twist a given $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting and such that the twisted splitting inherits irrationality from the original one. Moreover we need to control the change of area. All this is done in the present section. Section 3.3 is devoted to the inductive argument, which in turn will give the desired result in Section 3.4.

Definition. Let $\det(v, w)$ be the *signed area* of the parallelogram spanned by two vectors $v, w \in \mathbb{R}^2$: The absolute value of $\det(v, w)$ equals the euclidean area of the parallelogram and the sign is chosen to be positive if the pair (v, w) is positively oriented, negative otherwise.
Let $\mathrm{area}(C)$ be the area of one and, hence, the common area of all representing cylinders of the marked isometry class C.

Remark. The signed area of two vectors equals the determinant of the 2-by-2 matrix formed by these vectors as columns.

Let (T_1, T_2, C, w) be a $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting. Let $q \in \mathscr{L}$ be a quadratic differential which gives rise to this $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-surface. Using the notion of signed area we establish a criterion whether a union of parallelograms in T_1, T_2, C_1 and C_2 contains a cylinder.

Let v_1, v_2 be the holonomy vectors of simple closed curves in $T_1 \setminus \alpha$ and $T_2 \setminus \beta$ which join the initial point of the slit to itself, not intersecting the interior of the slit, and let v_c be the common holonomy of a pair of simple arcs in C_1 and C_2, joining the zero of q on one boundary component of the cylinder to the second zero on the other boundary component when imbedded into the $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-surface. For instance, in Figure 8 the images in the $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-surface of these curves and arcs may be the straight segments of the bottom line. If the images of the curves and arcs with holonomy v_1, v_2 and v_c concatenate to the core curve of a cylinder, they must have compatible orientations: $\det(v_j, w) > 0$ for all $j \in \{1, 2, c\}$ or $\det(v_j, w) < 0$ for all $j \in \{1, 2, c\}$. The interior of the parallelogram spanned by v_j and w is isometrically embedded in the respective torus or cylinder, $j \in \{1, 2, c\}$, therefore $|\det(v_i, w)| \leq \mathrm{area}(T_i)$, $i \in \{1, 2\}$, and $|\det(v_c, w)| \leq \mathrm{area}(C)$.

Conversely, suppose that v_1, v_2, v_c are three vectors in the Euclidean plane such that the just mentioned conditions on area and orientation are satisfied. The area condition assures that in the image of C_1 and C_2 there are simple arcs γ_{c1} and γ_{c2} with holonomy v_c, joining the zero of q on one boundary component of the respective cylinder to the second zero on the other boundary component, and that in the image of T_i, $i \in \{1, 2\}$, there is a simple closed curve γ_i with holonomy

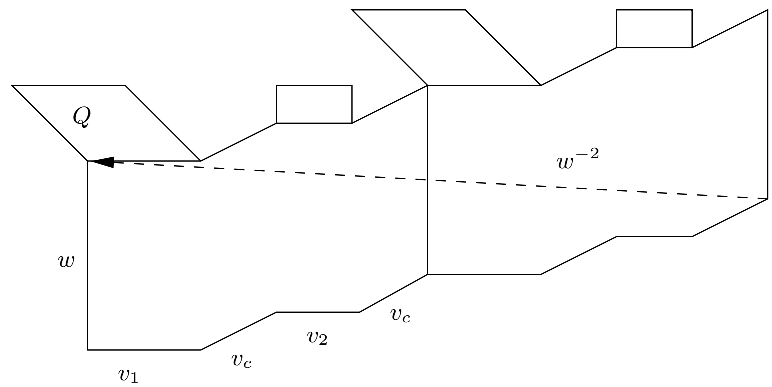

FIGURE 10. A splitting vector w and its twist w^{-2}. The horizontal cylinder Q is not affected.

v_i joining the initial/terminal point of the slit to itself. The orientation condition allows us to concatenate γ_1, γ_{c1}, γ_2 and γ_{c2} to a simple closed curve γ. For given $k \in \mathbb{N}$, we get four new simple closed curves α_+^k, α_-^k, β_+^k and β_-^k with common holonomy by Dehn twisting α_+, α_-, β_+ and β_- along the simple closed curve γ for k times. If we are lucky, each of the twisted curves can be realized by a single saddle connection with holonomy $w^k = w + k(v_1 + v_2 + 2v_c)$.

Lemma 3.4. *Each of the twisted simple closed curves α_+^k, α_-^k, β_+^k and β_-^k is realized by a single saddle connection if w and w^k lie on the same side of v_1, v_2 and v_c, i.e. if all signed areas $\det(v_j, w)$ and $\det(v_j, w^k)$, $j \in \{1, 2, c\}$, are positive or all are negative.*

Proof. Suppose $\det(v_j, w) > 0$ for all $j \in \{1, 2, c\}$. The inequality $\det(v_j, w^k) > 0$ for the twisted case will be referred to as (I_j), $j \in \{1, 2, c\}$. If $\det(v_1, v_2 + 2v_c) > 0$ we consider α_+^k, otherwise α_-^k. Assume $\det(v_1, v_2 + 2v_c) > 0$.

First, assume $k < 0$. As the $\mathrm{SL}(2, \mathbb{R})$-action preserves area, we may assume that w is vertical, pointing upwards, and v_1 is horizontal, pointing to the right. The Inequalities (I_1), (I_2) and (I_c) tell us that, as in Figure 10, the vector w^k (which is the holonomy of α^k) is above the lower boundary, hence the only possibility to hit the image of a zero of q is by reaching (or by crossing) the upper boundary. Moreover, if w^k crosses the upper boundary once, it will not come back from above. Let $\pi_v(x)$ denote the vertical component of a vector x. We examine Inequality (I_1) more closely:

$$\begin{aligned} 0 < \det(v_1, w^k) &= \det(v_1, w) + k \det(v_1, v_2) + 2k \det(v_1, v_c) \\ &= |v_1| \pi_v(w) + k|v_1| \big(\pi_v(v_2) + 2\pi_v(v_c)\big), \end{aligned}$$

therefore

$$\pi_v(w) > -k\pi_v(v_2 + 2v_c).$$

This implies that w^k does not cross the upper boundary and therefore does not hit the image of a zero of q beside at its endpoints. Hence α_+^k is realized by a single saddle connection, and so is α_-^k as of the action of the hyperelliptic involution.

Now assume $k > 0$. After applying the $\mathrm{SL}(2, \mathbb{R})$-action we may assume w to be vertical, pointing upwards, and $v_2 + 2v_c$ to be horizontal, pointing to the right. This causes $\pi_v(v_1)$ to be negative since $\det(v_1, v_2 + 2v_c) > 0$. Again, the Inequalities (I_1), (I_2) and (I_c) tell us that w^k is above the lower boundary, and, if it crosses the upper one, it will stay above. The Inequalities (I_2) and (I_c) lead to

$$\begin{aligned} 0 < \det(v_2 + 2v_c, w^k) &= \det(v_2 + 2v_c), w) + k \det(v_2 + 2v_c, v_1) \\ &= |v_2 + 2v_c|\pi_v(w) + k|v_2 + 2v_c|\pi_v(v_1), \end{aligned}$$

thus

$$\pi_v(w) > -k\pi_v(v_1).$$

As above, α_+^k is realized by a single saddle connection, and so is α_-^k.

In a similar manner we conclude for $\det(v_1, v_2 + 2v_c) < 0$ and for $\det(v_j, w) < 0$ and $\det(v_j, w^k) < 0$, $j \in \{1, 2, c\}$.

The same reasoning applies to β_+ and β_- instead of α_+ and α_-. $\square$

Remark. The new splitting is a $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splitting again: The two cylinders in the new splitting are isometric as flat surfaces, and the isometry may be chosen to preserve the horizontal direction.

Up to now we established conditions that allow us to construct new $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splittings from old ones by applying Dehn twists along wisely chosen simple closed curves. The next step is to find conditions for Dehn twists to preserve the property of irrationality of the $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splittings. We make a definition first.

Definition. Let (T_1, T_2, C, w) be a $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splitting. By Λ_1 and Λ_2 we will denote lattices in $\mathbb{R}^2$ which define the marked isometry classes of the tori T_1 and T_2 as quotients $\mathbb{R}^2/\Lambda_i$. By Λ_c we will denote the lattice generated by w and by the holonomy of a simple arc connecting the singularities on the two boundary components of one cylinder of the $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splitting.

The following proposition helps us to find Dehn twists preserving irrationality of $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splittings.

Proposition 3.5. *Let (T_1, T_2, C, w) be a splitting such that the slope of w is irrational in T_1, and let v_1, v_2 and v_c be holonomy vectors such that each α_+^k, α_-^k, β_+^k and β_-^k is realized by one saddle connection and gives a $2T_{fix}2C$-splitting $\big(T_1(k), T_2(k), C(k), w^k\big)$ for at least three different $k = k_1, k_2, k_3$. Then at least one of the three splitting vectors w^k is irrational in the respective torus $T_1(k)$.*

Remark. The proposition is symmetric with respect to T_1 and T_2 in the sense that one of the three splitting vectors is irrational in $T_2(k)$, too. However, this k may be different from the k that we get for T_1.

Proof. Let Q be a maximal cylinder in T_1, disjoint from w, whose core curve has holonomy v_1, see Figure 10. Let γ_0 be the diagonal from the lower left corner to the upper right corner in the rectangular image of Q in Figure 10. The simple straight segment γ_0 concatenates with α_+ to a simple closed curve in T_1. Let v_0 be the holonomy of γ_0. Then Λ_1, the lattice of T_1, is generated by the linearly independent vectors $v_0 + w$ and v_1. The vectors v_0 and v_1 are linearly independent, too.

We claim: The vector w is a scalar multiple of an element in Λ_1 (*rational* for short) if and only if w is rational in the lattice Δ generated by v_0 and v_1.
Indeed, for $c \in \mathbb{R} \setminus \{0\}$ and $(a, b) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ we have two equivalent equations $(1 + ac)w = c\big(a(v_0 + w) + bv_1\big)$ and $w = c(av_0 + bv_1)$. This proves the claim.

Now, T_1 and $T_1(k)$ share the same cylinder Q. By the same argument, w^k is rational in $\Lambda_1(k)$, the lattice of $T_1(k)$, if and only if w^k is rational in Δ.

To prove the proposition, suppose that the vector w^k were rational in $T_1(k)$ for all $k \in \{k_1, k_2, k_3\}$. Then the three w^k are parallel to elements in Δ. Let Δ^* be the lattice generated by w and $v_1 + v_2 + 2v_c$. As Δ and Δ^* share the three directions w^k which are not parallel to each other, Δ and Δ^* are isogenous and they share all possible directions (c.f. [McM05], proof of Theorem 7.3). Thus w is parallel to an element in Δ and w is rational in T_1, a contradiction. $\square$

Thus a tupel of holonomy vectors gives rise to twisted $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-surfaces if conditions on the signed area of parallelograms spanned by these vectors are fulfilled. If additionally we can iterate the twist for several times, irrationality will be preserved in at least one twisted splitting. We want to single out tupel which meet the conditions.

Definition. We call the triple (v_1, v_2, v_c) of holonomy vectors *good partners* (with respect to w), if

$$\begin{aligned}
4|\det(v_1, v_2)| &< \frac{1}{9}\min\{|\det(v_1, w)|, |\det(v_2, w)|\},\\
4|\det(v_1, v_c)| &< \frac{1}{9}\min\{|\det(v_1, w)|, |\det(v_c, w)|\},\\
4|\det(v_2, v_c)| &< \frac{1}{9}\min\{|\det(v_2, w)|, |\det(v_c, w)|\}
\end{aligned}$$

and all signed areas $\det(v_j, w)$ have the same sign, $j \in \{1, 2, c\}$.

Remark. An easy computation shows that good partners fulfill the conditions for Lemma 3.4 with $|k| \leq 9$.

Corollary 3.6. *If (v_1, v_2, v_c) are good partners with respect to w in an irrational $2T_{fix}2C$-splitting (T_1, T_2, C, w), then at least one of the nine twists w^k (with $k \in \{1, \ldots, 9\}$ or $k \in \{-1, \ldots, -9\}$) leads to another irrational $2T_{fix}2C$-splitting.*

Proof. Suppose that the conditions of Corollary 3.6 are satisfied. By the above remark every w^k with $|k| \leq 9$ is the splitting vector of a $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting. Proposition 3.5 tells us that each of the three triples (w^1, w^2, w^3), (w^4, w^5, w^6) and (w^7, w^8, w^9) contains a splitting vector which is irrational in the respective torus $T_1(k)$. Let these vectors be w^{k_0}, w^{k_1} and w^{k_2}, where each k_l satisfies $3l + 1 \leq k_l \leq 3l + 3$. Applying Proposition 3.5 in (T_1, T_2, C, w) again, this time with respect to the triple $(w^{k_0}, w^{k_1}, w^{k_2})$ and to the torus T_2, we get at least one splitting which is irrational in $T_2(k)$, too, $k \in \{k_0, k_1, k_2\}$. This proves the corollary. $\square$

We established conditions on holonomy vectors to give rise to new irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings if the initial $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting is irrational. This allows us to build sequences of $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings. Convergence of the splitting directions is easy to achieve, hence in order to apply the Masur-Smillie criterion we have to control the change of area between a give $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting and the $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting that we get by twisting. The following lemma enables us to control the change of area in terms of the signed area of parallelograms.

Lemma 3.7. *Let (T_1, T_2, C, w) be a $2T_{\text{fix}}2C$-splitting of the flat metric defined by $q \in \mathscr{L}$ and let (T_1', T_2', C', w') be obtained by twisting k times. The change of area between the two tori indexed by* 1 *is estimated by*

$$\operatorname{area}(T_1 \Delta T_1') \leq 2|\det(v_1, w)| + |k|\big(|\det(v_1, v_2)| + 2|\det(v_1, v_c)|\big).$$

For $\operatorname{area}(T_2 \Delta T_2')$ *the indices* 1 *and* 2 *change positions.*

Proof. Let Q be as in the proof of Proposition 3.5. We have $Q \subset T_1 \cap T_1'$, hence $T_1 \Delta T_1' \subset (T_1 \setminus Q) \cup (T_1' \setminus Q)$. The areas on the right hand side are easily computed: $\operatorname{area}(T_1 \setminus Q) = |\det(v_1, w)|$ and $\operatorname{area}(T_1' \setminus Q) = |\det(v_1, w')|$. Using the formula $w' = w + k(v_1 + v_2 + 2v_c)$ the first statement follows. The second statement is immediate as the construction is symmetric in T_1 and T_2. □

3.3. **The inductive process.** In the previous section we constructed new irrational $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splittings from old ones by Dehn twisting along a simple closed curve. This curve is the core curve of a flat cylinder inside the union of some parallelograms in the flat metric. The parallelograms have to meet conditions on their signed area. This section is devoted to find such parallelograms, hence to find a suitable cylinder to apply Dehn twists. Using Ratner's theorem we are able to make the area of the parallelograms as small as we want. This together with the results in Section 3.2 combines to a construction respecting irrationality of $2\mathrm{T}_{\text{fix}}2\mathrm{C}$-splittings. In Section 3.4 we apply this construction inductively to find non-ergodic directions. In order to handle the inductive step we need a lemma that gives information about orbit closures.

The following notations will be helpful: Let $G = \mathrm{SL}(2, \mathbb{R})$ and let N be the unipotent subgroup of upper triangular matrices. We will look at the action of N on triples of unimodular lattices. This action is given by the diagonal action of N which we denoted by N_Δ. For $s \in \mathbb{R}$, let $G_s = \{(g, (n_s)^{-1} g n_s) : g \in G\}$ be the twisted diagonal of G, where $n_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$. Two lattice $\Lambda_A \subset \mathbb{R}^2$ and $\Lambda_B \subset \mathbb{R}^2$ are said to be *strongly non-commensurable* if there isn't any $s \in \mathbb{R}$ such that Λ_A and $n_s \Lambda_B$ are commensurable.

Proposition 3.8 (Ratner-style theorem)**.** *Let Λ_C be the unimodular standard lattice. Let Λ_A and Λ_B be unimodular lattices, neither of them containing a horizontal vector. If Λ_A and Λ_B are strongly non-commensurable then $\overline{N_\Delta(\Lambda_A, \Lambda_B, \Lambda_C)} = (G \times G \times N)(\Lambda_A, \Lambda_B, \Lambda_C)$. Otherwise $\overline{N_\Delta(\Lambda_A, \Lambda_B, \Lambda_C)} = (G_s \times N)(\Lambda_A, \Lambda_B, \Lambda_C)$, where $s \in \mathbb{R}$ is such that Λ_A and $n_s \Lambda_B$ are commensurable.*

Proof. The first case is Corollary 5.3 in [HLM08] which is based on a theorem of McMullen in [McM07]. This theorem uses Ratner's theorem ([Rat95]).

The second case can be proved as follows: Ratner's theorem tells us that we can write $\overline{N_\Delta(\Lambda_A, \Lambda_B, \Lambda_C)} = H(\Lambda_A, \Lambda_B, \Lambda_C)$ for some $H < G \times G \times G$. By a theorem of McMullen ([McM07], Theorem 2.6) we know that $\pi_{1,2}(H) = G_s$ under the projection $\pi_{1,2}$ to the two first factors and $\pi_{1,3}(H) = G \times N$ under the projection $\pi_{1,3}$ to the first and third factor, therefore $H < G_s \times N$. To see the other inclusion let $(g, (n_s)^{-1} g n_s, n) \in G_s \times N$. As the image of H under $\pi_{1,3}$ equals $G \times N$ and as $(g, n) \in G \times N$, we know that $(g, g^*, n) \in H$ for some $g^* \in G$. The first projection gives $\pi_{1,2}\big((g, g^*, n)\big) \in G_s$, hence $g^* = (n_s)^{-1} g n_s$ and therefore $(g, (n_s)^{-1} g n_s, n) \in H$. □

Now we are ready to establish the inductive step. Recall the definition of the lattices Λ_1, Λ_2 and Λ_c as the lattices defined by a $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting (T_1, T_2, C, w). Making use of the N_Δ-action we can construct new $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings from a given $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting (T_1, T_2, C, w). We sketch the idea first. If the splitting vector w is horizontal, it is not affected by the N_Δ-action. As of the G-factor in the closure of the N_Δ-orbit of $(\Lambda_1, \Lambda_2, \Lambda_C)$, in the closure of the N_Δ-orbit we can find points with arbitrarily small parallelograms, hence points with arbitrarily small parallelograms in the N_Δ-orbit itself. The action of $\mathrm{SL}(2,\mathbb{R})$ on flat metrics preserves area, thus parallelograms of small area can be found in the initial $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting. We make this precise:

Lemma 3.9. *Given an irrational $2T_{fix}2C$-splitting (T_1, T_2, C, w) and $\varepsilon > 0$, there exists an irrational $2T_{fix}2C$-splitting (T_1', T_2', C', w') with small change of direction $|\angle(w, w')| < \varepsilon$ and such that the change of area is smaller than ε.*

To prove the lemma we basically look for good partners to get twisted splittings. The main difficulty is to keep track of small signed areas. Hence the formal proof is technical.

Proof. Recall from linear algebra $|\det(u_1, u_2)| = |\sin(\angle(u_1, u_2))| \cdot |u_1||u_2|$ for $u_1, u_2 \in \mathbb{R}^2$. Rotations preserve angles, lengths and areas. Without loss of generality we assume w to be horizontal. Note that w is fixed by the N-action. Irrationality of the $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting implies that neither Λ_1 nor Λ_2 contains horizontal vectors. Let $\varepsilon' > 0$ be small. We consider two cases.

First, let Λ_1 and Λ_2 be strongly non-commensurable. Proposition 3.8 tells us that in this case $\overline{N_\Delta(\Lambda_1, \Lambda_2, \Lambda_c)} = (G \times G \times N)(\Lambda_1, \Lambda_2, \Lambda_c)$. Thus the horizontal direction in Λ_c is fixed, and any area preserving linear map may be applied to Λ_1 and Λ_2. We use this to find good partners. Choose $(\Lambda_1^*, \Lambda_2^*, \Lambda_c^*) \in (G \times G \times N)(\Lambda_1, \Lambda_2, \Lambda_c)$ and $(v_1^*, v_2^*, v_c^*) \in (\Lambda_1^*, \Lambda_2^*, \Lambda_c^*)$ with $|\det(v_c^*, w)| = \mathrm{area}(C)$ and $\angle(v_c^*, w) = \pi/2$, and such that $\angle(v_i^*, w)$ is arbitrarily close to $\pi/2$ for $i \in \{1, 2\}$. Note that $\angle(v_i^*, v_c^*)$ is arbitrarily close to zero, $i \in \{1, 2\}$. Let $|v_1^*| = |v_2^*| = 1$. If we change v_j^*, $j \in \{1, 2, c\}$, we always mean to make a new choice of the three lattices in the orbit closure and afterwards to make a new choice of the lattice points.
Look at the areas $|\det(v_1^*, v_2^*)|$ and $|\det(v_i^*, v_c^*)|$, $i \in \{1, 2\}$. The angle condition on v_1^* and v_2^* guarantees $|\angle(v_1^*, v_2^*)|$ to be arbitrarily small, hence $|\det(v_1^*, v_2^*)|$ can be made arbitrarily close to zero. On the other hand, $|\det(v_1^*, w)|$ and $|\det(v_2^*, w)|$ are arbitrarily close to $|v_1^*||w|$ and $|v_2^*||w|$, both greater than zero, hence

$$|\det(v_1^*, v_2^*)| < \varepsilon' \min\bigl(|\det(v_1^*, w)|, |\det(v_2^*, w)|\bigr). \tag{3}$$

In addition, $|\det(v_i^*, v_c^*)| = |\sin(\angle(v_i^*, v_c^*))| \cdot |v_i^*||v_c^*| = |\sin(\angle(v_i^*, v_c^*))| \cdot |v_i^*| \frac{\mathrm{area}(C)}{|w|}$ is close to zero, too, and $|\det(v_c^*, w)| = \mathrm{area}(C)$. Therefore the inequalities

$$|\det(v_i^*, v_c^*)| < \varepsilon' \min\bigl(|\det(v_i^*, w)|, |\det(v_c^*, w)|\bigr) \tag{4}$$

hold for $i \in \{1, 2\}$.

Before we proceed with the next step, we will consider the other case. Let Λ_1 and $n_s\Lambda_2$ be commensurable for some $s \in \mathbb{R}$. We want to find $(v_1^*, v_2^*, v_c^*) \in (\Lambda_1^*, \Lambda_2^*, \Lambda_c^*) \in (G_s \times N)(\Lambda_1, \Lambda_2, \Lambda_c)$ fulfilling Inequalities (3) and (4). Suppose v_1^*, $n_s v_2^*$ and v_c^* are parallel vectors. We stick to almost horizontal vectors. The directions of $n_s v_2^*$ and v_2^* are nearly the same in this case. As a first calculation we get

$$\begin{aligned}|\det(v_2^*, v_c^*)| &= |\sin(\angle(v_2^*, v_c^*))| \cdot |v_2^*||v_c^*| \\ &= |\sin(\angle(v_2^*, v_c^*))| \cdot |v_2^*| \frac{\operatorname{area}(C)}{|w|} \frac{1}{|\sin(\angle(v_c^*, w))|} \qquad \text{and} \\ |\det(v_2^*, w)| &= |\sin(\angle(v_2^*, w))| \cdot |v_2^*||w|.\end{aligned}$$

Let x be the horizontal and y be the vertical coordinate of $n_s v_2^*$, thus we can write $v_2^* = (x - sy, y)^t$. Recall that $n_s v_2^*$ and v_c^* are parallel, that w is horizontal and that s is fixed and only depends on the lattices Λ_1 and Λ_2. We compute the quotient $|\det(v_2^*, v_c^*)|/|\det(v_2^*, w)|$ and write the sines in terms of x and y:

$$\begin{aligned}\frac{|\det(v_2^*, v_c^*)|}{|\det(v_2^*, w)|} &= \frac{\operatorname{area}(C)}{|w|^2} \frac{|\sin(\angle(v_2^*, v_c^*))|}{|\sin(\angle(v_2^*, w))| \cdot |\sin(\angle(v_c^*, w))|} \\ &= \frac{\operatorname{area}(C)}{|w|^2} \frac{\left| \frac{|y|}{\sqrt{x^2 - 2sxy + (1+s^2)y^2}} - \frac{|y|}{\sqrt{x^2+y^2}} \right|}{\frac{|y|}{\sqrt{x^2 - 2sxy + (1+s^2)y^2}} \frac{|y|}{\sqrt{x^2+y^2}}} \\ &= \frac{\operatorname{area}(C)}{|w|^2} \left| \left(\frac{x^2}{y^2} + 1 \right)^{1/2} - \left(\frac{x^2}{y^2} - \frac{2sx}{y} + (1+s^2) \right)^{1/2} \right| \to 0\end{aligned}$$

for $x/y \to \infty$. Hence for $y < x/K$ with $K = K(s)$ very large, i.e. for $n_s v_2^*$ almost horizontal, we have

$$\frac{|\det(v_2^*, v_c^*)|}{|\det(v_2^*, w)|} < \varepsilon'.$$

Choose v_c^* satisfying $|\det(v_c^*, w)| = \operatorname{area}(C)$ and which makes a small angle to the horizontal: $|\tan(\angle(v_c^*, w))| < K$. The $(G_s \times N)$-actions enables us to find a v_2^* such that v_c^* and $n_s v_2^*$ are parallel and the above inequality holds. Shortening v_2^* without changing its direction assures $|\det(v_2^*, v_c^*)| < \varepsilon' |\det(v_c^*, w)|$, too. As Λ_1 and $n_s \Lambda_2$ are commensurable, the vectors v_1^* and $n_s v_2^*$ can be chosen to be parallel, too, thus v_1^* and v_c^* are parallel. This implies $|\det(v_1^*, v_c^*)| = 0$, and Inequalities (4) are fulfilled. Again we use the group action to shorten v_1^* and v_2^* in order to make $|\det(v_1^*, v_2^*)| \leq |v_1^*||v_2^*|$ small compared to $|\det(v_2^*, w)| = |\sin(\angle(v_2^*, w))| \cdot |v_2^*||w|$ as well as small compared to $|\det(v_1^*, w)| = |\sin(\angle(v_1^*, w))| \cdot |v_1^*||w|$. Inequality (3) holds, too.

In both cases – strongly non-commensurable lattices and commensurable up to N-action – we simultaneously shorten v_1^* and v_2^* to guarantee that $|\det(v_i^*, w)|$ and $|\det(v_i^*, v_c^*)|$ are less than ε', $i \in \{1, 2\}$. Thus the same holds for $|\det(v_1^*, v_2^*)|$. Approximate (v_1^*, v_2^*, v_c^*) in N_Δ-orbits of $(\Lambda_1, \Lambda_2, \Lambda_c)$ and note that N_Δ preserves area and leaves w invariant. Hence, there is a $(v_1, v_2, v_c) \in (\Lambda_1, \Lambda_2, \Lambda_c)$ fulfilling the Inequalities (3) and (4). By the N_Δ-action we can ensure that all signed areas $\det(v_j, w)$ have the same sign, $j \in \{1, 2, c\}$. For ε' small enough, these are good partners and thus give rise to a new irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting.

The lengths of vectors in a given lattice are bounded from below. Using the equality $|\sin(\angle(u_1, u_2))| = \frac{|\det(u_1, u_2)|}{|u_1||u_2|}$ and choosing a small value for ε' we see that $\max\big(|\angle(v_1, w)|, |\angle(v_1, v_2)|, |\angle(v_1, v_c)|\big)$ is small and therefore we have $|\angle(w, w')| < \varepsilon$. Furthermore, using Lemma 3.7, $\operatorname{area}(T_1 \Delta T_1') < \varepsilon$ and $\operatorname{area}(T_2 \Delta T_2') < \varepsilon$. $\square$

Remark. In fact, for the proof we constructed triples of good partners to get irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings. Every triple of good partners gives rise to two different new irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings, one for $k < 0$ and one for $k > 0$.

3.4. **Uncountably many non-ergodic directions.** We collected all the information needed to prove Theorem 3.1. This section contains the proof.

We outline the prove first. Let the flat metric defined by $q \in \mathscr{L}$ have an irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splitting (T_1, T_2, C, w). We build a rooted binary tree such that the rooted geodesics in the tree represent non-ergodic directions on $q \in \mathscr{L}$ with irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings. The vertices of the tree are directions of irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings of the flat metric defined by q and we add an oriented edge between two of them if the second one can be achieved by the first one using Lemma 3.9, where in each step we impose new conditions on angle and change of area. The Masur-Smillie criterion then gives the desired result.

Proof of Theorem 3.1. The root of the tree is the direction w of our initial splitting (T_1, T_2, C, w). For every direction w_n at level n, called parent, construct two different subsequent directions w_n^1 and w_n^2, called the children, such that these directions give rise to $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings with $|\angle(w_n, w_n^k)| < \varepsilon_n/4$ and $\operatorname{area}(T_1 \Delta T_1^k) < \varepsilon_n/4$, $k \in \{1, 2\}$, where $\varepsilon_n > 0$ is an arbitrary number smaller than all angles between any pair of parents and children constructed so far. In detail, given w_n and ε_n we apply Lemma 3.9 with $\varepsilon < \varepsilon_n/4$ to find two different children w_n^1 and w_n^2 with the desired properties. The resulting tree contains 2^n directions of irrational splittings at its n-th level. Given a geodesic in this rooted binary tree, we denote the splitting corresponding to the point on the geodesic in level n by (T_1^n, T_2^n, C^n, w_n).
The geodesics in this tree represent different converging sequences of splitting directions: The angles between two subsequent directions converge to zero. The series of the changes of area along a geodesic converges to a value smaller then the limit of the geometric series. To apply the Masur-Smillie criterion (Theorem 3.3) we have to show that in every geodesic in the tree of irrational $2\mathrm{T}_{\mathrm{fix}}2\mathrm{C}$-splittings the heights h_n perpendicular to the limiting direction of the splitting vectors w_n converge to zero. To establish this, we first note that for large n the inequality $h_{n+1} \leq 2|\det(w_n, w_{n+1})|/|w_{n+1}|$ holds. This follows from a basic calculation using $|\det(u_1, u_2)| = |\sin(\angle(u_1, u_2))| \cdot |u_1||u_2|$ and $|\angle(w_n, \theta_\infty)| \leq 2|\angle(w_n, w_{n+1})|$ for large n. Secondly, we note that all w_n point into different directions. As on any flat metric there are only finitely many saddle connection shorter than a given upper bound, the lengths of the splitting vectors w_n tend to infinity. Third ingredient is $|\det(w_n, w_{n+1})| \leq 9\big(\operatorname{area}(T_1^n) + \operatorname{area}(T_2^n) + 2\operatorname{area}(C^n)\big) < A < \infty$. The upper bound $A < \infty$ is independent of n as the sequence formed by the changes of area is bounded from above by the geometric series. Combining these three facts, we see that h_n converges to zero.

The Masur-Smillie criterion guarantees that there is a map from the set of infinite geodesic starting at the vertex corresponding to (T_1, T_2, C, w) to the set of non-ergodic directions on q. This map is injective by the sequence of angles we chose in the construction. Thus there is an uncountable number of non-ergodic directions. As there are at most countably many saddle connections, hence at most countably many non-minimal directions, we can find uncountably many minimal and non-ergodic directions. Theorem 3.1 is proven. □

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